LECTURE 1 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.1 Introduction to Random Variables

LEARNING OUTCOMES : At the end of the lesson students should be able to:
(a) Define random varible.

SET INDUCTION

* Explain what is variable.

Variable was defined as a characteristics or attribute that can assume different values. Various letters of alphabet, such as x, y, or z, are used to represent variables.

Example 1 : If a dice is rolled, a letter such as ‘x’ , can be used to represent the outcomes. Then the value x can assume is 1, 2, 3, 4, 5 or 6, corresponding to the outcomes of rolling a single dice.

Example 2 : If two coins are tossed, a letter ‘y’ can be used to represent the number of heads, in this case 0, 1 or 2.

Example 3 : If the temperature (T) at 8.00 am is 43^{o} and at noon it is 53^{o}, then the values of the temperature (T) assumes are said to be random, since they are due to various atmospheric conditions at the time the temperature was taken.

Since the variables are associated with probability, they are called **random variables.**

CONTENTS

Introduction to Random Variables

Consider an experiment of tossing simultaneously two fair coins once. The outcomes of the experiment can be listed as a sample space, ***S* = {*HH, HT, TH, TT*}**, where ***H* = Head**, ***T* = Tail** and can also be represented as a variable, say *X*. Let ***X* = number of tail appearing**. The possible outcomes can be tabulated as follows:

Possible outcomes	*TT*	*HT*	*TH*	*HH*
Number of tail, *X*	2	1	1	0

Therefore, the possible values of *X* are 0, 1 and 2. These value, 0, 1 and 2 are actually random values specified by the possible outcomes obtained from the experiment. **These random values are known as random variables**.

Definition

A **random variables** is a function that assigns a numerical value to each simple event in a sample

Usually random variables are denoted by capital letters such as X and Y and their corresponding small letters, x and y are used to represent a particular value of the random variable.

Example 1

If two coins are tossed and a letter ***X* is used to represent the number of heads.**

Example 2

If two dice are rolled and a letter ***Y* is used to represent the sum of the number shown on dice**,

Discrete Random Variable

A **discrete random variable** assumes values that can be counted. In other words, the consecutive values of a discrete random variable are separated by a certain gap. A random variable that assumes countable values is called a discrete random variable.

Some examples of discrete random variables are:

1. The number of cars sold at a dealership during a given month.
2. The number of employees working at a company.
3. The number of complaints received at the office of an airline on a given day.
4. The number of customers visiting a bank during any given hour.
5. The number of heads obtained in three tosses of a coin.

Continuous Random Variable

A random variable whose **values are not countable** is called a ***continuous random variable.*** A continuous random variable can assume any value over an interval or several intervals.

A random variable that can assume any value contained in one or more intervals is called a continuous random variable.

Because the number of values contained in any interval is infinite, the possible number of values that a continuous random variable can assume is also infinite. Moreover, we cannot count these values. Consider the life of battery with the maximum life of 200 hours. If X denote the life of a randomly selected battery, then X can assume any value in the interval 0 to 200. Consequently, X is a continuous random variable.

The following are a few examples of continuous random variables.

1. Salaries of workers in a factory.
2. Time taken by workers to learn a job.
3. Amount of water consumed by a family in each month of a year.
4. The height of students in a college.

LECTURE 2 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.2 Discrete Random Variables

LEARNING OUTCOMES : At the end of the lesson students should be able to:
(a) construct the probability distribution table and probability distribution function.

CONTENTS

Probability Distribution of a Discrete Random Variable

The list of all the possible values of random variable *X* with their corresponding probabilities is known as **discrete probability distribution**.

Suppose $x_1, x_2, ...x_k$ are the values of a ***discrete random variable*** X which are associated with the corresponding probabilities $P(X = x_1), P(X = x_2),....., P(X = x_k)$.
If $\sum_{i=1}^{k} P(X = x_i) = 1$, then X is known as a discrete random variable.

If *X* is a discrete random variable then

$$\boxed{0 \le \mathrm{P}(\mathrm{X} = \mathrm{x_i}) \le 1 \text{ and } \sum_{i=1}^{k} P(X = x_i) = 1}$$

The probability distribution can be presented in the form of table, function and graph.

Example 1

Construct the probability distribution function, if two fair coins are tossed.

Solution

Example 2

The probability distribution function of a discrete random variable X is given by $P(X = x) = k(4 - x)$, for $x = 0, 1, 2, 3$. Given that k is a constant, find the value of *k*.

Solution:

Example 3

A fair die is rolled. If *X* represents the number on die, show that X is a discrete random variable. Find $P(X > 3)$ and $P(1 < X \leq 5)$.

Solution :

Example 4

The probability distribution function of a discrete random variable X is given by $P(X = x) = kx^2$ for $x = 0,1,2,3,4$.

(a) Find the value of the constant k.

(b) Obtain the probability distribution of X.

(c) Find (i) $P(X = 1 \; or \; X = 4)$

(ii) $P(|X - 2| < 1)$

Solution:

Example 5

A bag contains four pieces of red towels and three yellow towels. The towels are to be drawn at random one by one without replacement from the bag until a piece of red towel is obtained. If X is the total number of towels drawn from the bag:

(a) obtain the probability distribution of X.

(b) find $P(1 < X \leq 3)$.

Solution:

LECTURE 3 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.2 Discrete Random Variables

LEARNING OUTCOMES : At the end of the lesson students should be able to:

b) find the cumulative distribution function and its median.
c) find the probabilities from
   i) probability distribution table or function
   ii) cumulative distribution function.

CONTENTS

Cumulative Distribution Function

The corresponding cumulative frequencies are obtained from a frequency table by summing all the frequencies up to a particular value. The same idea is extended to probability distribution ie the corresponding cumulative probabilities are obtained by summing all the probabilities up to a particular value. The resulting function is known as ***cumulative distribution function***.

(a) Discrete Random Variable

If X is a discrete random variable with probability distribution function $P(X=x)$ for $x = x_1, x_2, \ldots, x_n$, then the cumulative distribution function of *X*, denoted by $F(x)$, is given by

$$\boxed{\begin{aligned} F(x) &= P(X \le x) \\ &= \sum_{x=x_1}^{x_n} P(X=x) \end{aligned}}$$

THE MEDIAN, M IS THE VALUE WHEN $P(X \le m) = 0.5$ OR $F(m) = 0.5$

Example 1

The probability distribution function of a discrete random variable X is given in the table below:

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(a) Find the cumulative distribution function, $F(x)$.

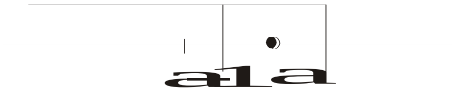
(b) Sketch the graph $F(x)$.

(c) Find the median

Solution:

Note: In general,

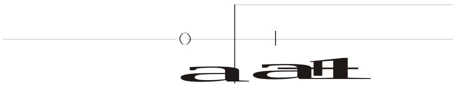
1. $P(X = a) = P(X \le a) - P(X \le a-1)$



$= F(a) - F(a-1)$

Eg: $P(X = 3) = F(3) - F(2)$

2. $P(X > a) = 1 - P(X \le a)$



$= 1 - F(a)$

Eg: $P(X > 4) = 1 - F(4)$

3. $P(a < X \le b) = P(X \le b) - P(X \le a)$

$= F(b) - F(a)$

Eg: $P(5 < X \le 7) = F(7) - F(5)$

4. $P(a < X < b) = P(X \le b-1) - P(X \le a)$

$= F(b-1) - F(a)$

Eg: $P(4 < X < 6) = F(5) - F(4)$

5. $P(a \le X \le b) = P(X \le b) - P(X \le a-1)$

$= F(b) - F(a-1)$

Eg: $P(2 \le X \le 6) = F(6) - F(1)$

Example 2

The cumulative distribution function $F(x)$ of a discrete random variable X is shown as below:

X	1	2	3	4	5
$F(x)$	0.2	0.32	0.67	0.9	1

Find:

(a) $P(X = 3)$

(b) $P(X > 2)$

(c) $P(2 \le X \le 4)$

(d) the median.

Solution:

Example 3

The cumulative distribution function of a discrete random variable X is given in the table below:

X	1	2	3	4	5
$F(x)$	0.05	0.3	0.6	0.75	1

Find the probability distribution function of X.
Solution:

LECTURE 4 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.2 Discrete Random Variables.

LEARNING OUTCOMES : At the end of the lesson, students should be able to :

d) Calculate the expectation and variance of discrete random variables.

CONTENT

Expectation of X (Discrete Random Variable)

Expectation or the expected value of a random variable X (discrete or continuous) is the mean of X and is denoted by μ or E(X).

If X is a discrete random variable with probability distribution function P(X=x),then

$$E(X) = \sum^{n} x_i P(X = x_i)$$

Example 1

The probability distribution of a discrete random variable X is given as follows.

x	1	2	3	4	5
P(X = x)	0.1	0.3	0.3	0.2	0.1

Calculate E(X).

Solution

Example 2

A discrete random variable X has the following probability distribution:

X	1	2	t
P(X=x)	0.1	0.2	0.7

Find the value of t if E(*X*) = 4.

Solution

Example 3

A discrete random variable X can only take the values 2 and 3, and has expectation 2.6. Find the probabilities *P(X=2)* and *P(X=3)*.

Solution

Expectation of any function of X

The definition of expectation , $E(X) = \sum_{all\ x} xP(X = x)$ can be extended to any function of random variable such as 3X , X^2 , $(X - 2)^3$ etc.
If X is a ***discrete random variable*** with probability distribution function $P(X=x)$, then the expectation of any function of X , g(X) , is given by

$$E[g(X)] = \sum_{all\ x} g(x)P(X = x).$$

In particular , when g(X) = X^2,

$$E(X^2) = \sum_{all\ x} x^2 P\ (X = x)$$

Important properties of expectation

1. $E(a) = a$, where a is constant.
2. $E(aX) = a\ E(X)$, where a is a constant.
3. $E(aX + b) = a\ E(X) + b$, where a and b are constants.

Notes:
The above properties of expectation are true for both discrete and continuous random variables.

Example 4

The probability distribution of a discrete random variable X is given in the table below:

X	1	2	3
P(X = x)	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

Find:
(a) $E(2)$ (b) $E(X)$
(c) $E(5X)$ (d) $E(5X + 2)$
(e) $E(X^2)$ (f) $E(5X^2 + 2)$

Solution

Example 5

X is a discrete random variable and k is a constant.

If $E(3X+k) = 26$ and $E(2k - X) = 3$, find the value k and $E(X)$.

Solution

Variance of Discrete Random Variable

Variance of a discrete random variable X is denote by Var (X) or σ^2 and is defined as

$$\text{Var}(X) = E[(X - \mu)^2]$$

where $\mu = E(X)$.

Notes:

1. $$\begin{aligned} Var(X) &= E[(X - \mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2[E(X)]^2 + [E(X)]^2, \text{ as } \mu = E(X) \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

$$\boxed{Var(X) = E(X^2) - [E(X)]^2}$$

2. The standard deviation of a random variable is denoted by σ. As such,

$$\boxed{\text{Standard deviation of } X = \sqrt{\text{Var (X)}}}$$

3 The variance of X is always positive

Important properties of variance

1. $Var(a) = 0$, where a is constant.
2. $Var(aX) = a^2\, Var(X)$, where a is a constant.
3. $Var(aX + b) = a^2\, Var(X)$, where a and b are constants.

Notes:
The above properties of variance are true for both discrete and continuous random variables.

Example 6

The probability distribution of a discrete random variable X is given as follows:

x	1	2	3
$P(X=x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

Calculate :
(a) $E(X)$
(b) $E(X^2)$
(c) $Var(X)$

Solution:

Example 7

The probability distribution function of a discrete random variable X is given as follows:

x	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{16}$

Find

(a) *Var (X)*

(b) *Var (X – 2)*

(c) *Var (2X + 3)*

(d) *Var (5 – 8X)*

Solution

LECTURE 5 OF 8

TOPIC : RANDOM VARIABLE

SUBTOPIC : 9.3 Continuous Random Variables

OBJECTIVES : At the end of the lesson students are able to:

(a) Find the

i) probability density function.

(b) Determine the probabilities from a probability density function

SET INDUCTION:

- Ask a few students about their weight or height.
- Ask them how they got the figure.
  (Answer: By measuring)
- Ask them whether they can count their weight or their height?
  (Answer: No, they have to measure it)
- Explain: They can count the numbers of cars, numbers of student etc.
  (These are discrete variables)
- Explain : Data that can't be counted but can be obtained by measuring are continuous variables
- Ask for more examples

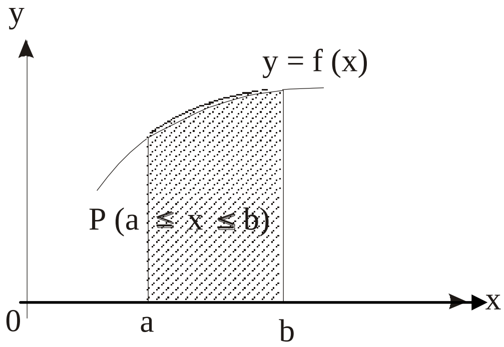
CONTENT:

Probability density function

Continuous random variables are theoretical representations of continuous variables such as height, mass or time. A continuous random variable is specified by its ***probability density function*** which is written as *f(x).* This function is defined over the range $(-\infty,+\infty)$.

If the probability density function *f(x)* ≥ 0 for all *x* and $\int_{-\infty}^{\infty} f(x)dx = 1$, then X is known as a continuous random variable.

Note:



1. If X is a continuous random variable with probability density function *f(x)*, then $P(a \le X \le b) = \int_a^b f(x)dx$. Notice that $\int_a^b f(x)dx$ is in fact the area enclosed by the curve $y = f(x)$, *x*-axis, the line x= a and x=b.
2. As $P(X = k) = 0$, where k is constant, then $\mathrm{P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)}$. Unlike discrete distribution, we do not need to worry about inequalities.

Example 1

Given $f(x) = \begin{cases} \frac{3}{7}x^2, & 1 \le x \le 2 \\ 0, & otherwise \end{cases}$

(a) Show that the function is the probability density function (p.d.f)

(b) P(X< $\frac{3}{2}$)

Solution:

Example 2

A continuous random variable X has probability density function
$f(x) = kx^3, \; for \; 0 \le x \le 4$.

Find: (a) the value of k (b) $P(1 \le X \le 3)$

(c) $P(X > 2)$

Solution:

Example 3

The continuous random variable X has probability density function

$$f(x)=\begin{cases} k(x+1) & , \quad 0 \le x < 2 \\ 2k & , \quad 2 \le x \le 3 \\ 0 & , \quad otherwise \end{cases}$$

(a) Find the value of the constant k.

(b) Sketch the probability density function.

(c) Find *P (1.5<X<2.5).*

(d) Find *P (X >1.8).*

Solution:

Example 4

A continuous random variable X has probability density function

$$f(x) = \begin{cases} k\sqrt{x}, & 0 \le x \le 1 \\ 0, & \textit{otherwise} \end{cases}$$

Find :
(a) the value of the constant k.
(b) P(0.3 ≤ X<0.6)

Solution :

LECTURE 6 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.3 Continuous Random Variables

OBJECTIVES : At the end of the lesson students are able to:

(a) ii) Find the cumulative distribution functions.

CONTENT:

(b) Cumulative Distribution Function

The probability of a continuous random variable is obtained by integration, so is its cumulative distribution function F(x).

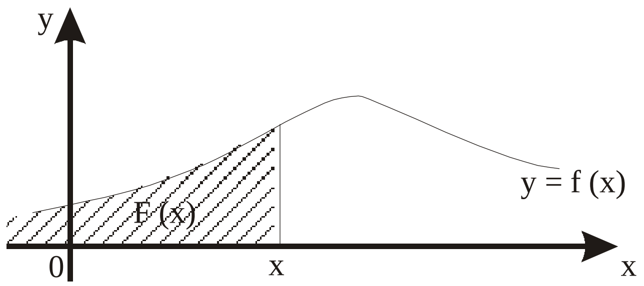
If X is a ***continuous random variable*** with probability density function $f(x)$ for $-\infty < x < \infty$, then the cumulative distribution function of X is given by

$$\boxed{F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx}$$

Notice that the link with the formula for the discrete random variables, P(x) is replaced by $f(x)$ and Σ by $\int$.

Notes:

1. F(x) is in fact given by the area under the curve y = $f(x)$ from $-\infty$ up to x as indicated by the shaded region.

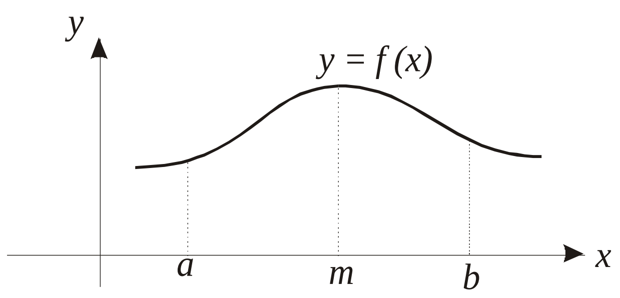


2. F $(-\infty)$ = 0 and F(∞) =1
3. P (X<a) = P(X ≤ a) = F(a)
4. P (a ≤ X ≤ b) = P (a < X ≤ b)
   = P (a ≤ X < b)
   = P (a < X < b)
   = F (b) – F (a)

Notes:

There is one value that is particularly important in the cumulative distribution function. As learnt in the earlier chapter, the median was defined as the middle value, with exactly half the data lying below that value and half lying above that value. In the context of a probability density function, the median will be that point below which the area is 0.5 and above which the area is also 0.5 .In other words, ***the median divides the area under the curve into halves.***

If the median is m, then P(X$\leq$m) = $\int_a^m f(x)dx = 0.5$, which means F (m) = 0.5.



Example 1

X is a continuous random variable with probability density function $f(x) = \frac{1}{8}x$,

where $0 \leq x \leq 4$.

(a) Find the cumulative distribution function F(x).

(b) Sketch the graph y = F(x).

(c) Calculate P(2 < x < 3) .

(d) Find the median F(x).

Solution:

Example 2

The probability density function of a continuous random variable X is defined as

$$f(x)=\begin{cases} \dfrac{3}{16}(x+2)^2 & , \quad -2 \le x < 0 \\ \dfrac{3}{16}(x-2)^2 & , \quad 0 \le x < 2 \\ 0 & , \quad otherwise \end{cases}$$

(a) Obtain the cumulative distribution function , F(x).

(b) Sketch the graphs f and F.

(c) Find P(X > 1).

(d) Find the median of F(x).

Solution:

Example 3

A continuous random variable X has the following probability density function

$$f(x)=\begin{cases} \dfrac{x}{3}-\dfrac{2}{3} & , \quad 2 \le x < 3 \\ \dfrac{1}{3} & , \quad 3 \le x < 5 \\ 2-\dfrac{1}{3}x & , \quad 5 \le x < 6 \\ 0 & , \quad \textit{otherwise} \end{cases}$$

Find:

(a) F(x)

(b) P $(2 \le \text{X} \le 3.5)$

(c) P $(\text{X} \ge 5.5)$

Solution:

Example 4

X is a continuous random variable with probability density function

$f(x) = \frac{1}{4}(4-x)$, for $1 \le x \le 3$. Find:

(a) the cumulative distribution function, $F(x)$,

(b) the median m .

Solution:

LECTURE 7 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.3 Continuous Random Variables

OBJECTIVES : At the end of the lesson students are able to:

c) find the probability density function from the cumulative distribution function and vice versa.

CONTENT:

The Probability density function from the cumulative distribution function.
Since the cumulative distribution function is the **integral** of the probability density function, it follows that the probability density function of a continuous random variable is the **derivative** of the cumulative distribution. That is,

$$\text{F}(x) = \frac{\text{d}}{\text{dx}}\left[\text{F(x)}\right]$$

(f (x) is the derivative of F(x))

Example 1

The continuous random variable X has a cumulative distribution function given by

$$\text{F}(x) = \begin{cases} 0 \ , & x < 0 \\ \dfrac{x^3}{27} \ , & 0 \le x \le 3 \\ 1 \ , & x > 3 \end{cases}$$

(a) Find the probability density function, f(x).
(b) Sketch the function y = f (x).

Solution:

Example 2

The cumulative distribution function, F(x), of a continuous random variable X is defined as follows :

$$F(x)=\begin{cases} 0, & x<-2 \\ \frac{1}{12}(2+x), & -2\le x<0 \\ a+\frac{x}{6}, & 0\le x<4 \\ b+\frac{x}{12}, & 4\le x<6 \\ 1, & x\ge 6 \end{cases}$$

(a) Find the values of the constants a and b.
(b) Find the probability density function, $f(x)$.
(c) Calculate $P(3\le X<5)$.

Solution :

Example 3

The cumulative distribution function of x is given by

$$F(x)=\begin{cases} 0 & , \quad x<0 \\ 2x-x^2 & , \quad 0\le x\le 1 \\ 1 & , \quad x>1 \end{cases}$$

a) Find the probability density function of x.

b) Hence, find $P(X \ge \frac{1}{2})$

Solution:

LECTURE 8 OF 8

TOPIC : 9.0 RANDOM VARIABLES

SUBTOPIC : 9.3 Continuous Random Variables.

LEARNING

OUTCOMES : At the end of the lesson, students should be able to :

d) Calculate the expectation and variance of discrete random variables.

CONTENT

Expectation of X

Expectation or the expected value of a continuous random variable X with probability density function f(x), is the mean of X and is denoted by μ or E(X).

$$E(X) = \int_{-\infty}^{\infty} xf(x)\ dx$$

Example 1

The probability density function of a continuous random variable X is defined as follows:

$$f(x) = \begin{cases} \dfrac{x}{3}, & 0 \le x < 1 \\ \dfrac{1}{3}, & 1 \le x < 3 \\ \dfrac{1}{3}(4-x), & 3 \le x \le 4 \\ 0, & otherwise \end{cases}$$

Find E(X).

Solution

Expectation of any function of X

If X is a ***continuous random variable*** with probability density function *f(x)*, then the expectation of any function of X, *g(X)* is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \,.$$

In particular , when g(X) = X^2,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx \,.$$

Important properties of expectation

1. *E(a) = a*, where a is constant.
2. *E(aX) = a E(X)*, where a is a constant.
3. *E(aX + b) = a E(X) + b*, where a and b are constants.

***The above properties of expectation are true for both discrete and continuous random variables.**

Example 2

A continuous random variable X has the following probability density function

$$f(x)=\begin{cases}\dfrac{2}{7}(x-1) & , \quad 2\le x<5\\ 0 & , \quad otherwise\end{cases}$$

Find:

(a) $E(X+1)$ (b) $E(2X)$

(c) $E(10-X)$ (d) $E(3X^2-5)$

Solution

Variance of function of Continuous Random Variables

Variance of X

Variance of a continuous random variable X is denote by Var (X) or σ and is defined as

Var $(X) = E\,[(X - \mu)^2]$ where $\mu = E(X)$.

Notes:

1. $$\begin{aligned} Var\,(X) &= E\,[\,(X - \mu\,)^2\,] \\ &= E\,[\,X^2 - 2\,\mu X + \mu^2\,] \\ &= E\,(\,X^2) - 2\mu E(X) + \mu^2 \\ &= E\,(X^2) - 2\,[\,E(X)]^2 + [E(X)]^2 \text{ , as } \mu = E(X) \\ &= E\,(X^2) - [E(X)]^2 \end{aligned}$$

$$\boxed{Var\,(X) = E\,(X^2) - [\,E(X)\,]^2}$$

2. The standard deviation of a random variable is denoted by σ. As such,

$$\boxed{\text{Standard deviation of } X = \sqrt{\text{Var (X)}}}$$

Example 3

X is a continuous random variable with probability density function defined as follows:

$$f(x) = \begin{cases} \frac{4}{3}(1-x), & 0 \le \mathrm{x} < \frac{1}{2} \\ \frac{4}{3}x \quad , & \frac{1}{2} \le x < 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Calculate $Var\,(X)$.

Solution

Important properties of variance

1. $Var\ (a) = 0$, where a is constant.
2. $Var\ (aX) = a^2\ Var\ (X)$, where a is a constant.
3. $Var\ (aX + b) = a^2\ Var(X)$, where a and b are constants.

The above properties of variance are true for both discrete and continuous random variables.

Example 4

X is a continuous random variable with probability density function $f(x)$.

If $\int_{-\infty}^{\infty} x^2 f(x)dx = 29$ and $Var\ (3X) = 36$,

find the mean of X.

Solution:

Example 5

The continuous random variable X has probability density function f(x) given by

$$f(x) = \frac{3}{4}\left(1 + x^2\right) \ , \quad 0 \le x \le 1$$

If $E(X) = \mu$ and $Var(X) = \sigma^2$, find $P(|X - \mu| < \sigma)$.

Solution: