LECTURE 1 OF 5

TOPIC : 8.0 PROBABILITY

SUBTOPIC : 8.1 Events and Probability

LEARNING OUTCOMES : At the end of the lesson students should be able to:
(a) understand the concept of experiments, outcomes, events, sample spaces and random selections

SET INDUCTION

Suppose that there are a blue pen, a red pen and a black pen. What is the chance of choosing a black pen?

Answer: 1/3 (this is known as probability)

CONTENTS

Experiment, outcomes, sample space and event

Suppose that a process that could lead to two or more different outcomes is to be observed and there is certainty beforehand as to which outcome will occur. Below are some examples.

1. A coin is tossed.
2. A die is rolled.
3. A card is drawn from pack of 52 cards.
4. A ball is taken out from a box containing 20 balls with different colour.

Each of these examples involves a ***random experiment***.

Definition

An **experiment** is a situation involving chance or probability that leads to results called outcomes.

If a coin is thrown, the result will be either a “head” or a “tail”.
If a die is rolled, the result will be one of the numbers 1,2,3,4,5 or 6.

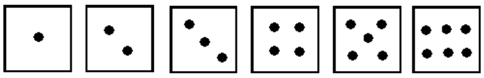
In each case, the different possible outcomes are called ***basic outcomes***.
The set of <u>all these outcomes</u> which exhausts all the possibilities is called the ***sample space*** of the random experiment.

Definition

The possible outcomes of a random experiment are called the **basic outcomes**, and the set of all basic outcomes is called the **sample space**.

Notice that basic outcomes are defined in such a way that no two outcomes can occur simultaneously; moreover, the random experiment must necessarily lead to the occurrence of one of the basic outcomes. The symbol S will be used to denote the sample space.

Example 1
A die is rolled.



The sample space, S = { 1, 2, 3, 4, 5, 6 }, $n(S) = 6$

Example 2
Two coins are tossed.
The sample space,
S = { HH, HT, TH, TT}, $n(S) = 4$

<u>Definition</u>

> An **event** is a set of basic outcomes from the sample space, and is said to occur if the random experiment gives rise to one of its constituent basic outcomes.

Example 3 (refer example 1)
S = { 1, 2, 3, 4, 5, 6 }, $n(S) = 6$
Let A be the event getting value greater than three
B be the event getting an odd value

Example 4 (refer example 2)
S = { HH,HT,TH,TT }, $n(S) = 4$
Let X be the event getting one head
Y be the event getting at least one head

PROBABILITY

<u>Definition</u>

If we have a number of equiprobable events, the probability of a particular outcome is define as

$$\frac{\text{the number of equiprobable favourable outcome}}{\text{the total number of equiprobable outcome}}$$

The probability that an event, A, occur is defined as the number of ways in which A can happen expressed as a fraction of the number of ways in which all equally likely events, including A, occur.

The probability of an event A occurring is denoted by P(A), where

$$P(A) = \frac{\text{number of successful outcomes}}{\text{total number of all possible outcomes}}$$

Or

$$P(A) = \frac{\text{No. of members of } \{A\}}{\text{No. of members of } \{\text{possibility space}\}} = \frac{n(A)}{n(S)}$$

Probability Of An Event

$$P(A) = \frac{\text{The Number Of Ways Event A Can Occur}}{\text{The total number Of Possible Outcomes}}$$

As A is subset of S or {possibility space}, the numerator of this fraction is always less than, or equal to, the denominator, so for any event A,

$$\boxed{0 \le P(A) \le 1}$$

If P(A) = 1 ,this means that the event is an absolute certainty.
If P(A) = 0 this means that the event is an absolute impossibility.

For example, if one ball is taken from a bag containing only red balls:
P(ball is red) = 1 and P(ball is blue) = 0.

A probability is a number from 0 to 1. If we assign a probability of 0 to an event, this indicates that this event *never* will occur. A probability of 1 attached to a particular event indicates that this event *always* will occur. What if we assign a probability of .5? This means that it is just as likely for the *event to occur* as for the event *to not occur.*

THE PROBABILITY SCALE

```
+------------------------+------------------------+
0                        .5                        1
event never      event and "not event"        always
will occur      event are likely to occur   will occur
```

Example 5

Find the probability of getting a head when you flip a coin.
Solution

Example 6

Find the probability of obtaining a number greater than 4 on a single toss of a die.
Solution

VENN DIAGRAMS, TREE DIAGRAM AND CONTIGENCY TABLE

Example 7
In a certain group of 100 students, 54 studied calculus, 69 studied physics and 35 studied both. If one of these students is selected at random, find the probability that

a) The student took calculus or physics
b) The student did not take either of two subjects.
c) The student took physics but not calculus.

Solution

Example 8
A coin is tossed twice. Find the probability of getting at least one head.

Solution

Example 9
Two dice are tossed. Find the probability of

a) the sum of two numbers are 8.
b) The sum of two numbers are prime numbers.

Solution

LECTURE 2 OF 5

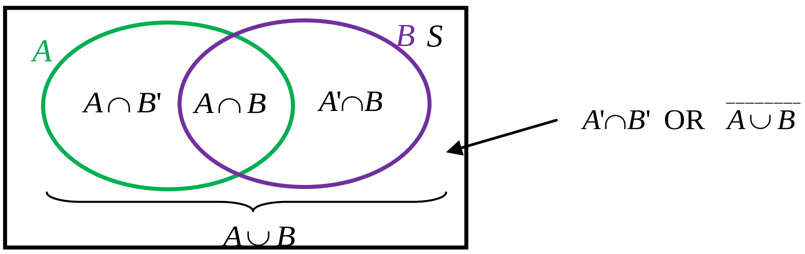
TOPIC : 8.0 PROBABILITY

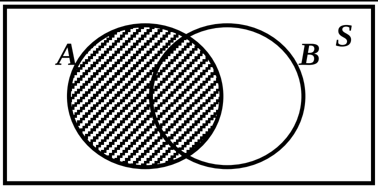
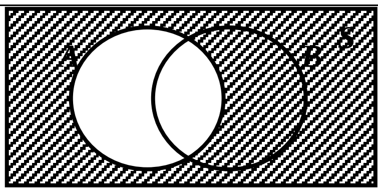
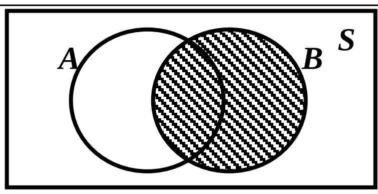
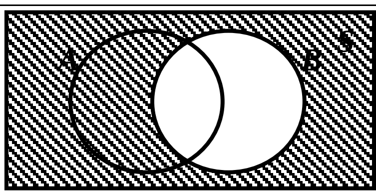
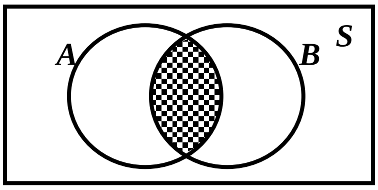
SUBTOPIC : 8.1 Events and Probability

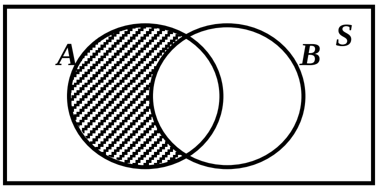
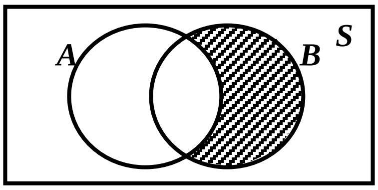
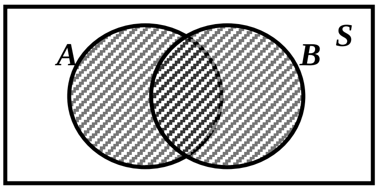
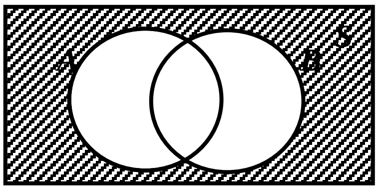
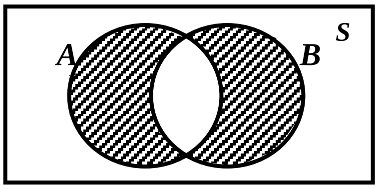
LEARNING OUTCOMES : At the end of the lesson students should be able to:

a) State the basic laws of probability
b) Find the probability of an event
c) Determine probabilities of the intersections and union of two events.

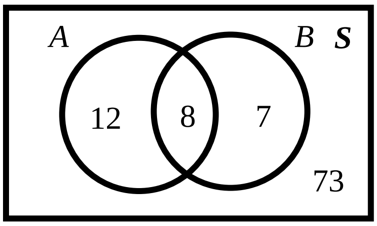
VENN DIAGRAMS



A		*A* happens
A'		*A* does not happen
B		*B* happens
B'		*B* does not happen
$\boldsymbol{A\cap B}$		*A* and *B* happen simultaneously (both *A* and *B* happen)

$A \cap B'$		*A* happens but *B* does not happen(only *A* happens)
$A' \cap B$		*B* happens but *A* does not happen (only *B* happens)
$A \cup B$		*A* happen or *B* happen or both happen (at least one happen)
$A' \cap B'$		Both *A* and *B* do not happen
$(A \cap B') \cup (A' \cap B)$		Only one happen (either *A* or *B* but not both)

Example 1

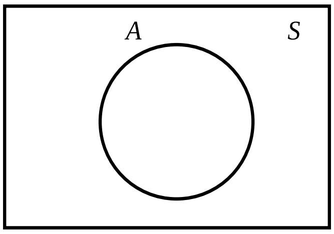


Find

$P(A)$, $P(A')$, $P(B)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$,
$P(A \cap B')$, $P(A' \cap B)$, $P[(A \cap B') \cup (A' \cap B)]$

Solution

Important Result in Probability



i) $P(S) = 1$

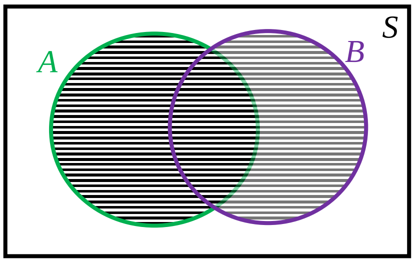
ii) $0 \le P(A) \le 1$

iii) $P(A') = 1 - P(A)$ or $P(\overline{A}) = 1 - P(A)$

A' is the complement of an event A

b) (i) Probability of two events *A* or *B* occurring

$P(A$ or B or both$) = P(A) + P(B) - P(A$ and $B)$, can be denoted as $P(A \cup B)$,



$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

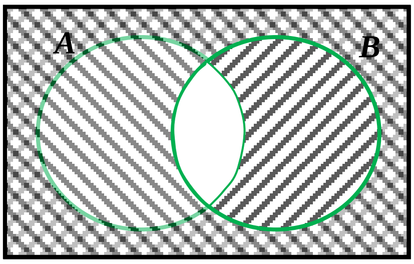
Prove $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$\begin{aligned} P(A \cup B) &= \frac{n(A \cup B)}{n(S)} \\ &= \frac{n(A) + n(B) - n(A \cap B)}{n(S)} \\ &= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

De Morgan Rule

$$\boxed{\begin{aligned} P(A' \cup B') &= P(A \cap B)' \\ P(A' \cap B') &= P(A \cup B)' \end{aligned}}$$

$P(A \cap B') = P(A) - P(A \cap B)$



Example 2

Given A and B are 2 events where $P(A) = \frac{1}{3}$, $P(B) = \frac{5}{9}$ and $P(A \cap B) = \frac{1}{6}$. Find

(a) $P(A \cup B)$
(b) $P(A \cap B')$
(c) $P(A' \cup B')$
(d) $P(A' \cap B')$

Solution

Example 3

If P(A) = 0.3, P(B) = 0.4 and $P(A \cap B) = 0.1$, find

(a) $P(A \cup B)$

(b) $P(\overline{A \cup B})$

(c) $P(A \cup \overline{B})$

(d) $P(\overline{A} \cap B)$

Solution

Example 4

Records showed that 80% of the drivers who are summoned for various traffic offences are man drivers. While 17% of all drivers are below 30 years old, 13% are man drivers who are below 30 years. If a driver who is summoned is randomly selected, what is the probability that the driver is a man or below 30 years old?

Solution

Example 5

A survey is conducted on a group of workers comprising production operators, administrative officers and security guards. The survey is to determine the total working hours in a week.

	Production operator (A)	Administrative officer (B)	Security guard (C)	Total
<40 hrs (D)	63	21	4	88
50–70 hrs (E)	46	14	10	70
>70 hrs (F)	87	8	17	112
	196	43	31	

One of the workers in the survey is randomly selected. Based on the information provided, calculate the probability

(a) The workers being a production operator.

(b) The workers who work between 50–70 hours.

(c) The workers being an administrative officer and working greater than 70 hours.

(d) The workers being a security guard working less than 40 hours.

Solution

LECTURE 3 OF 5

TOPIC : 8.0 PROBABILITY

SUBTOPIC : 8.1 Events and Probability

LEARNING OUTCOMES : At the end of the lesson students should be able to:
a) Find the probability of an event
b) Determine probabilities of the intersection and union of two events

PROBABILITY OF MUTUALLY EXCLUSIVE EVENTS

- Two or more events are said to be mutually exclusive if they cannot occur at the same time.
- In other words, events that have no outcomes in common are said to be mutually exclusive events or disjoint events.

Examples of Mutually Exclusive

- A six-sided die is rolled once.
  Event A : rolling an odd number
  Event B : rolling an even number
  Events A and B are mutually exclusive, because they cannot both happen.

- If two events A and B are mutually exclusive, then the probability of the occurrence of A or B is the sum of their individual probabilities. Two events A and B are called mutually exclusive if they cannot occur at the same time.

Additional Probability Rules for two mutually exclusive events,

$$\mathbf{P(A \cup B) = P(A) + P(B)}$$

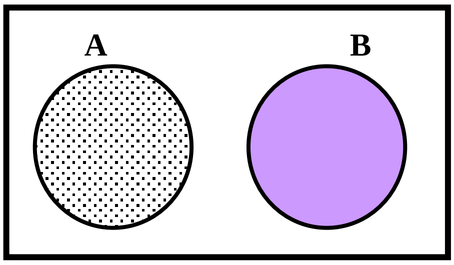


Figure: Mutually exclusive event.

Since neither event A nor event B can occur simultaneously thus the probability of A and B occurring at the same time is zero. This can be denoted as $P(A \cap B) = 0$.

Example 1
C and D are two events where $P(C) = 0.1$, $P(D) = 0.2$ and $P(C \cup D) = 0.3$.
(a) Determine whether C and D are two mutually exclusive events.
(b) Find $P(C')$ and $P(C' \cap D')$

Solution

Example 2

A bag contains 4 red marbles, 2 white marbles and 8 black marbles. What is the probability that a marble picked from the bag at random is either red or white?

Solution

Example 3

The following is the information that is received from the final exam result in one college for Mathematics and Account course. According to the analysis that is done, the result is:

600 students passed the Account course.
300 students passed the Mathematics course.
175 students passed both course.
50 students failed both course.

From the information above,

(a) How many of the college students took the Mathematics and Account courses?
(b) Calculate the probability of students who passed both courses.
(c) Find the probability of students who did not pass in Mathematics.

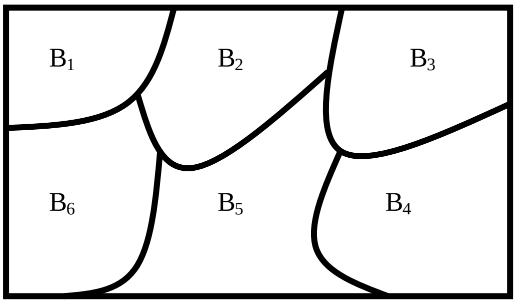
Solution

Definition:

If n events $X_1, X_2, X_3, \ldots, X_n$ are mutually exclusive, then
$P(X_1 \cup X_2 \cup \ldots \cup X_n) = P(X_1) + P(X_2) + \ldots P(X_n)$

Outcomes are exhausted if they are combined to be the entire sample space, or equivalently, if at least one of the outcomes must occur whenever the experiments performed.

Definition:
If $B_I \cup B_2 \cup \ldots \cup Bn = Sn,$ the entire sample space, then the events $B_I, B_2, \ldots, Bn$ are referred to as exhaustive events.



All events are mutually exclusive
$P(B_I \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6) = P(S) = 1$

Example 4
A selection committee held a meeting to choose a student leader. Those who attended the meeting were 20 first semester students, 30 second semester students, 10 third semester students and 20 fourth semester students. Find the probability of the semester from which the student leader would be chosen if first semester students are not qualified to be a student leader.

Solution

Solving Problem Involving Events and Probability

Example 5

One student feels that the probability he will get a grade D in a statistics course is $\frac{1}{8}$ and the probability he will fail in that course is $\frac{1}{16}$. What is the probability he will get a grade better than D?

Solution

Example 6

4 letters are chosen randomly from the word COMPUTER. Find the probability

(a) all the four letters chosen are consonant

(b) the letter C must be chosen

(c) the letters M and P must be chosen simultaneously

Solution

Example 7

Three red marbles, four yellow marbles and two green marbles are arranged in one row on a table. Find the probability

(a) all the four yellow marbles must be next to each other

(b) all the four yellow marbles must not be arranged next to each other

(c) the green marbles must be in the first and last position of the row

Solution

LECTURE 4 OF 5

TOPIC : 8.0 PROBABILITY

SUBTOPIC : 8.2 Conditional Probability

LEARNING OUTCOMES : At the end of the lesson students should be able to:

a) Determine the conditional probability and identify independent events.

CONDITIONAL PROBABILITY

Definition:
For events A and B in an arbitrary sample space S, we define the conditional probability A given B by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad ; \ P(B) \neq 0$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad ; \ P(A) \neq 0$$

P (A / B) is read "the probability of A, given B"

Example 1

Given that P(A) = $\frac{1}{4}$; P(B) = $\frac{1}{3}$; P(A $\cup$ B)= $\frac{5}{12}$

Find: a) P(A $\cap$ B) b) P(B/A)

Solution

Example 2

Given that P(A) = $\frac{1}{3}$, P(B) = $\frac{1}{4}$ and P(A $\cup$ B)= $\frac{1}{2}$

Find : a) P(A/B) b) P(B/A)

c) P(A $\cap$ B') d) P(A/B')

Solution

Example 3

In a college, 12% of the students are left-handed, 15% of the students wear glasses and 3% are both left-handed and wear glasses.

a) Given that a student wears glasses, find the probability that the student is left-handed.
b) What is the probability that a left-handed student also wears glasses?

Solution

Example 4

Thirty Mathematics professors out of 100 who are examined were found to be over weight (W). Ten of them had high blood pressure (H). Only four of the professors who were not over weight had high blood pressure.

Find the probability that a Mathematics professor will not have high blood pressure if he is not overweight.

Solution

Independent Events

A is the event of the 'result of spinning a fair coin' and B is the event of the 'number obtained when a pair dice is rolled'. These events are unrelated. Obtaining a head on the coin will not influence the outcome on the die. Such events are said to be independent.

From Conditional Event;

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore \quad P(A \cap B) = P(B) \,.\, P(A|B)$$

If events A and B are independent, it means that the outcome of one event does not affect the outcome of the other, then

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

So, we arrive at the **'and'** rule for the independent events

$$P(A \text{ and } B) = P(A) \,.\, P(B)$$

$$P(A \cap B) = P(A) \,.\, P(B)$$

Example 1

A, B and C are three events such that A and B are independent whereas A and C are mutually exclusive. Given $P(A) = 0.4$ and $P(B) = 0.2$, $P(C) = 0.3$ and $P(B \cap C) = 0.1$.

Find a) $P(A \cup B)$ b) $P(C \mid B)$ c) $P(C \mid A')$

Solution

Example 2

Probability that Siti is still alive in 20 years is 0.7, and probability that Lan will be alive in 20 years is 0.9. If we assume independence for both, what is the probability that neither will be alive in 20 years?

Solution

Example 3

A Mathematics puzzle is given to 3 students, Amin, Bernard and Chong. From past experience, it is found that the probabilities that Amin, Bernard and Chong will get the correct solution are 0.65 , 0.6 and 0.55 respectively.

If three of them attempt to solve the puzzle without consulting each other, find the probability that:

a) The puzzle will be solved correctly by all of them
b) Only one of them will get the correct solution

Solution

Example 4

There are 60 students in the sixth form of a certain school. Mathematics is studied by 27 of them, Biology by 20 and 22 students study neither Mathematics nor Biology.

a) Find the probability that a randomly selected student studies both Mathematics and Biology.
b) Find the probability that a randomly selected student studies Mathematics but does not study Biology.

A student is selected at random,

c) Determine whether the event ‘studying mathematics’ is statistically independent of the event ‘studying Biology’.

Solution

LECTURE 5 OF 5

TOPIC : 8.0 PROBABILITY

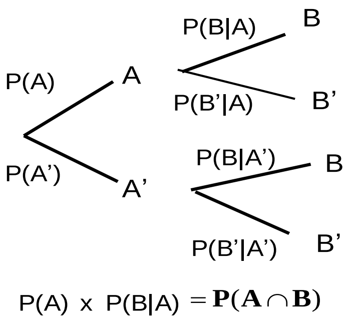
SUBTOPIC : 8.2 Conditional Probability

LEARNING OUTCOMES : At the end of the lesson students should be able to:

a) Use of Venn diagrams, tree diagrams and table of outcomes to solve probability problems.

Tree diagrams

A tree diagram consists of a number of branches that illustrate all the possible outcomes of a sequence of experiments or events where each event can occur in a finite numbers of ways.



Note : To find the probabilities , you must multiply along the branches and add between the branches.

Example 1

A bag contains 8 balls, 5 yellow and 3 red. A ball is drawn at random without replacement. Another draw is made. What is the probability :

(a) that both balls are red?

(b) that the balls are not of the same colour?

Solution

Example 2

There are 12 red balls and 8 green balls in a bucket. Two balls are taken out in sequence without replacement. By using a tree diagram, find the probability that

(a) the first ball is red

(b) the second one is red if the first is red

(c) the second one is red if the first is green

(d) the second one is red

(e) the first one is red if the second is red

Solution

TABLE

Example 3

A survey was made among 200 persons whether or not to impose the death penalty on peddlers of pornographic VCD. The table below summarizes the responses.

	Favour	**Oppose**	**Neutral**	**Total**
Man	70	30	50	150
Lady	30	10	10	50
Total	100	40	60	200

Find the probability that a person selected at random from these 200 persons is

a) A man or is in favour to impose the death penalty
b) In favour to impose the death penalty or neutral
c) A lady and is opposed to impose the death penalty

Solution

Example 4

500 persons (male and female) were asked if they are in favour of or against capital punishment. Of the 300 males, 125 are in favour, whereas 145 females are against. If a person is selected at random from these 500 persons, find the following probabilities.

a) In favour

b) In favour given female

c) Male against

d) In favour or female

Are the events 'male' and 'in favour' independent? Are they mutually exclusive? Give explanations.

Solution

Example 5

If
$P(X) = 0.2, P(Y) = 0.5 \; and \; P(X \cap Y) = 0.1, find$

a) $P(X \cup Y)$ e) $P(\bar{X} \cap Y)$

b) $P(X \cup Y)'$ f) $P(\bar{X} \cup Y)$

c) $P(Y \backslash X)$ g) $P(Y \backslash \bar{X})$

d) $P(X \backslash Y)$ h) $P(\bar{X} \cap \bar{Y})$

Solution