1. Integration by part,

$$u = xe^x \quad \Rightarrow \quad \frac{du}{dx} = xe^x + e^x(1) = e^x(x+1)$$

$$dv = \frac{1}{(x+1)^2}, \Rightarrow \quad v = \int \frac{1}{(x+1)^2}\, dx = -\frac{1}{x+1}$$

$$\therefore \quad \int_0^1 \frac{xe^x}{(x+1)^2}\, dx = \left[xe^x\left(-\frac{1}{x+1}\right) - \int\left(-\frac{1}{x+1}\right)e^x(x+1)\, dx \right]_0^1$$

$$= \left[-\frac{xe^x}{x+1} + \int e^x\, dx \right]_0^1$$

$$= \left[-\frac{xe^x}{x+1} + e^x \right]_0^1$$

$$= \left(-\frac{e}{2} + e\right) - (0+1)$$

$$= \frac{e}{2} - 1$$

2. (a) Integration by part,

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{4x} \quad \Rightarrow \quad v = \int e^{4x}\, dx = \frac{e^{4x}}{4}$$

$$\therefore \quad \int xe^{4x}\, dx = \frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4}\, dx$$

$$= \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + c$$

(b) By substitution,

$$u = x - 2 \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$u = x - 2 \quad \Rightarrow \quad x = u + 2$$

$$\therefore \quad \int x\sqrt{x-2}\, dx = \int (u+2)\sqrt{u}\, du$$

$$= \int u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\, du$$

$$= \frac{2u^{\frac{5}{2}}}{5} + \frac{4u^{\frac{3}{2}}}{3} + c$$

$$= \frac{2}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + c$$

3. $\dfrac{dh}{dt} = -k\sqrt{h}$

$\int dh = \int -k\sqrt{h}\ dt$

$2\sqrt{h} = -kt + c$

When $t = 0, h = 9 \quad \Rightarrow \quad c = 6$

$\therefore \quad 2\sqrt{h} = -kt + 6$

When $t = 15, h = 4 \quad \Rightarrow \quad k = \dfrac{2}{15}$

$\therefore \quad 2\sqrt{h} = -\dfrac{2}{15}t + 6$

When $h = 4 \quad \Rightarrow \quad \dfrac{2}{15}t = 6$

$\therefore \quad t = 45$ minutes.

<u>Alternative:</u>

$\dfrac{dV}{dt} \propto \sqrt{h} \Rightarrow \dfrac{dV}{dt} = k\sqrt{h}$

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$k\sqrt{h} = 4\pi \times \dfrac{dh}{dt} \qquad \left\{ V = \pi r^2 h, \text{ where } r = 2 \Rightarrow \dfrac{dV}{dh} = 4\pi \right\}$

$\dfrac{4\pi}{\sqrt{h}}\dfrac{dh}{dt} = k$

$4\pi \int h^{-\frac{1}{2}}\ dh = \int k\ dt$

$8\pi\sqrt{h} = kt + c$

When $t = 0, \quad h = 9 \quad \Rightarrow \quad 8\pi\sqrt{9} = c \quad \Rightarrow \quad c = 24\pi$

$\Rightarrow \quad 8\pi\sqrt{h} = kt + 24\pi$

When $t = 15, \quad h = 4 \quad \Rightarrow \quad 8\pi\sqrt{4} = 15k + 24\pi \quad \Rightarrow \quad k = -\dfrac{8\pi}{15}$

$\therefore \quad 8\pi\sqrt{h} = -\dfrac{8\pi}{15}t + 24\pi$

When $h = 0 \quad \Rightarrow \quad 0 = -\dfrac{8\pi}{15}t + 24\pi$

$\Rightarrow \quad t = 45$

So, the tank empties in 45 minutes.

4. $\tan x = 2x \quad \Rightarrow \quad \tan x - 2x = 0$

Let $f(x) = \tan x - 2x$

$$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} - \frac{\pi}{2} = 1 - \frac{\pi}{2} = -0.571 < 0$$

$$f\left(\frac{\pi}{2}\right) = \tan\frac{\pi}{2} - \pi > 0 \qquad \left\{ \lim_{x \to \left(\frac{\pi}{2}\right)^-} \tan x = +\infty \right\}$$

$\therefore$ There is a root between $\frac{\pi}{4}$ and $\frac{\pi}{2}$.

Newton-Raphson method,

$$f(x) = \tan x - 2x \quad \Rightarrow \quad f'(x) = \sec^2 x - 2$$

Let $x_1 = \frac{\pi}{3}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{\pi}{3} - \frac{\tan\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{3}\right)}{\sec^2\left(\frac{\pi}{3}\right) - 2} = 1.228370$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.177632$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.166044$$

$$x_5 = 1.165562$$

$$x_6 = 1.165561$$

$\therefore$ The required root is 1.166 (to 3 d.p.).

5. $16x^2 + 9y^2 - 32x + 36y - 92 = 0$

$16(x^2 - 2x) + 9(y^2 + 4y) = 92$

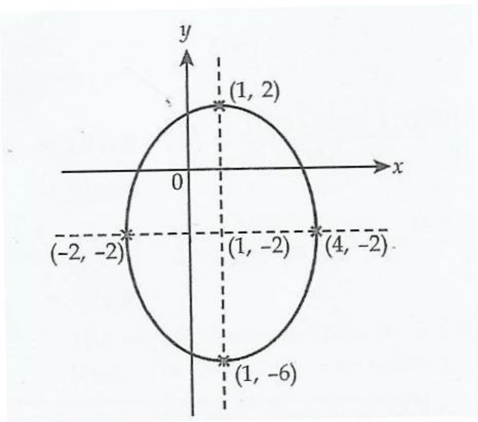
$16(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = 92 + 16 + 36$

$16(x-1)^2 + 9(y+2)^2 = 144$

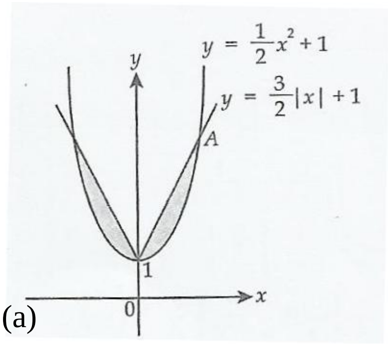
$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{16} = 1$ is an ellipse.

Centre $(1, -2)$; vertices $(1, -2 \pm 4)$

The graph of the ellipse,



6.



(a)

(a) Solving $y = \dfrac{x^2}{2} + 1$ and $y = \dfrac{3}{2}x + 1$ for coordinates of *A*:

$$\frac{x^2}{2} + 1 = \frac{3}{2}x + 1 \quad \Rightarrow \quad x = 3, \quad y = \frac{11}{2} \quad \Rightarrow \quad A\left(3, \frac{11}{2}\right)$$

The area is symmetrical about the *y*-axis.

$$\therefore \quad \text{Area} = 2\int_0^3 \left[\left(\frac{3}{2}x + 1\right) - \left(\frac{x^2}{2} + 1\right)\right] dx$$

$$= 2\int_0^3 \left[\frac{3}{2}x - \frac{x^2}{2}\right] dx$$

$$= 2\left[\frac{3}{4}x^2 - \frac{1}{6}x^3\right]_0^3$$

$$= 4\frac{1}{2} \text{ unit}^2$$

(b) Volume $= \pi\int_1^{\frac{11}{2}} x^2 \, dy - \text{volume of the cone}$

$$= 2\pi\int_1^{\frac{11}{2}} (y - 1) \, dy - \frac{1}{3}\pi(3)^2\left(\frac{11}{2} - 1\right)$$

$$= 2\pi\left[\frac{y^2}{2} - y\right]_1^{\frac{11}{2}} - \frac{27}{2}\pi$$

$$= 6\frac{3}{4}\pi \text{ unit}^3.$$