1. Let $x=\sqrt[3]{4}$

$x^3=4$

$x^3-4=0$

Let $f(x)=x^3-4$

$f(1)=1-4=-3<0$

$f(2)=2^3-4=4>0$

$\therefore$ The root is between 1 and 2.

$f'(x)=3x^2$

Using Newton-Rhapson formula,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

Since the root lies between $[1,2]$,

Let $x_0=1.5$

$$x_1=1.5-\frac{(1.5)^3-4}{3(1.5)^2}=1.59259$$

$$x_2=1.59259-\frac{(1.59259)^3-4}{3(1.59259)^2}=1.58742$$

$$x_3=1.58742-\frac{(1.58742)^3-4}{3(1.58742)^2}=1.58740$$

$$x_4=1.58740-\frac{(1.58740)^3-4}{3(1.58740)^2}=1.58740$$

$\therefore$ The root is 1.587. (3 decimal places)

2. Gradient of $y+x=9$ is -1.

$\therefore$ Gradient of the normal is 1.

The equation of the normal at (3,6), $y-6=1(x-3)$

$$y-x=3$$

Let the centre of the circle (h,k). Since its lies on $y-x=3$

The second curve $k-h=3$ ---------- (1)

The radius of the circle $\sqrt{(3-h)^2+(6-k)^2}=\sqrt{(5-h)^2+(2-k)^2}$

$$2k-h=4 \text{ ------------ (2)}$$

Solve (1) and (2): $k=1$

$$1-h=3$$

$$h=-2$$

$\therefore$ The centre is $(-2,1)$.

The radius is $\sqrt{(3+2)^2+(6-1)^2}=\sqrt{50}=5\sqrt{2}$

The equation of circle is $(x+2)^2+(y-1)^2=50$.

3. $x\dfrac{dy}{dx}-y=3x^2$

$$\frac{dy}{dx}-\frac{1}{x}y=3x$$

$$P(x)=-\frac{1}{x} \qquad Q(x)=3x$$

$$V(x)=e^{\int P(x)dx}$$

$$=e^{\int -\frac{1}{x}dx}$$

$$=e^{-\ln x}$$

$$V(x)=\frac{1}{x}$$

$$V(x)\cdot y=\int V(x)\cdot Q(x)dx$$

$$\frac{1}{x}\cdot y=\int\frac{1}{x}\cdot 3x\,dx$$

$$\frac{1}{x}\cdot y=\int 3\,dx$$

$$\frac{1}{x}\cdot y=3x+c$$

$$y=3x^2+cx$$

when $y=0$, $x=2$: $\quad 0=3(2)^2+c(2)$

$$c=-6$$

$$\therefore\ y=3x^2-6x$$

4. $y=e^x$

when $x=0$, $y=e^0=1$

As $x\to+\infty, y\to+\infty$

As $x\to-\infty, y\to 0$

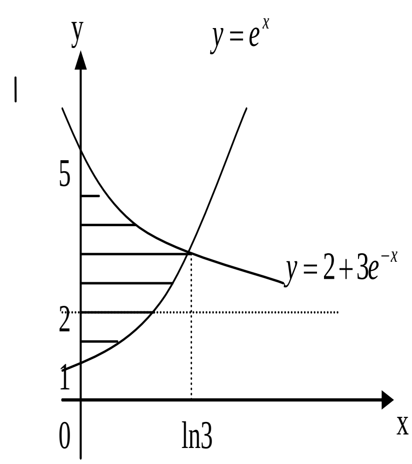
$y=2+3e^{-x}$

$$y=2+\frac{3}{e^x}$$

when $x=0,\quad y=2+\dfrac{3}{e^0}=5$

As $x\to+\infty, \dfrac{3}{e^x}\to 0$ and thus $y\to 2$

As $x\to-\infty, y\to+\infty$



$y=e^x$ ----------- (1)

$y=2+3e^{-x}$ ----- (2)

Substituting (1) into (2), $e^x = 2 + 3e^{-x}$

$$e^x = 2 + \frac{3}{e^x}$$

$$(e^x)^2 = 2e^x + 3$$

$$(e^x)^2 - 2e^x - 3 = 0$$

$$(e^x - 3)(e^x + 1) = 0$$

$$e^x = 3 \text{ or } e^x = -1$$

$$x = \ln 3$$

Hence, the x-coordinate of the point of intersection of the curves is $\ln 3$.

Area of the shaded region

$$= \int_0^{\ln 3} \left[\left(2 + 3e^{-x}\right) - e^x\right] dx$$

$$= \left[2x + 3\left(\frac{1}{-1}\right)e^{-x} - e^x\right]_0^{\ln 3}$$

$$= \left[2x - \frac{3}{e^x} - e^x\right]_0^{\ln 3}$$

$$= 2\ln 3 - \frac{3}{e^{\ln 3}} - e^{\ln 3} - \left(2(0) - \frac{3}{e^0} - e^0\right)$$

$$= 2\ln 3 - \frac{3}{3} - 3 - 0 + 3 + 1$$

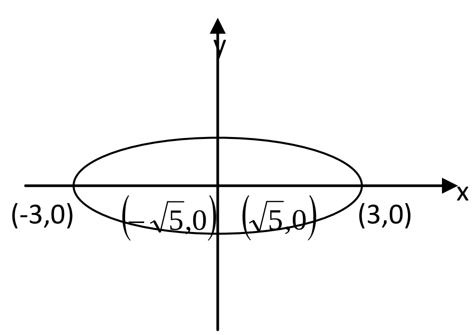
$$= 2.20 \text{ units}^2$$

5. a) $4x^2 + 9y^2 = 36$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$c^2 = 9 - 4 = 5$$

$$c = \sqrt{5}$$



$\text{Foci} = \left(\sqrt{5},0\right)$ and $\left(-\sqrt{5},0\right)$

b) $4x^2 + 9y^2 = 36$ ------------ (1)

Implicit differentiation: differentiate respect to x

$$8x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y}$$

At $\frac{dy}{dx} = \frac{2}{9}$, $\frac{2}{9} = -\frac{4x}{9y}$

$$2y = -4x$$

$y = -2x$ ---------- (2)

Substitute (2) into (1):

$$4x^2 + 9(-2x)^2 = 36$$

$$4x^2 + 9(4x^2) = 36$$

$$40x^2 = 36$$

$$x^2 = \frac{36}{40} = \frac{9}{10}$$

$$x = \pm\sqrt{\frac{9}{10}} = \pm\frac{3}{\sqrt{10}} = \pm\frac{3\sqrt{10}}{10}$$

when, $4\left(\frac{9}{10}\right) + 9y^2 = 36$

$$y^2 = \frac{18}{5}, \quad y = \pm 3\sqrt{\frac{2}{5}}$$

Coordinates of the curve when $\frac{dy}{dx}=\frac{2}{9}$ are $\left(\frac{3\sqrt{10}}{10},-3\sqrt{\frac{2}{5}}\right)$ and $\left(-\frac{3\sqrt{10}}{10},3\sqrt{\frac{2}{5}}\right)$.

6. a) $\int \frac{3x}{e^x}dx=\int 3xe^{-x}dx$

$u=3x \qquad dv=e^{-x}dx$

$du=3dx \qquad v=-e^{-x}$

$$\int 3xe^{-x}dx=(3x)\left(-e^{-x}\right)-\int -e^{-x}3dx$$

$$=-3xe^{-x}+3\int e^{-x}dx$$

$$=-3xe^{-x}-3e^{-x}+c$$

$$=-3e^{-x}[x+1]+c$$

b) $$\frac{x+2}{(1-x)(x^2+2)}=\frac{A}{1-x}+\frac{Bx+C}{x^2+2}$$

$$x+2=A(x^2+2)+(Bx+C)(1-x)$$

when $x=1,\ 3=3A$

$A=1$

By comparing the coefficients,

$x^2:\quad 0=A-B$

$B=1$

$x^0:\quad 2=2A+C$

$C=2-2$

$C=0$

$$\therefore \frac{x+2}{(1-x)(x^2+2)}=\frac{1}{1-x}+\frac{x}{x^2+2}$$

$$\int \frac{x+2}{(1-x)(x^2+2)}\,dx = \int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{2x}{x^2+2}dx$$

$$= -\ln|1-x| + \frac{1}{2}\left[\ln|x^2+2|\right] + C$$