LECTURE 1 OF 7
TOPIC : **5.0 VECTORS**
SUBTOPIC : **5.1 Vectors in Three Dimensions**
LEARNING OUTCOMES : At the end of the lesson, students should be able to:

(a) determine the types of vectors
(b) perform addition and scalar multiplication

5.1 Vectors In 3-Dimensions

The Cartesian coordinate for space are often called *rectangular coordinate* .

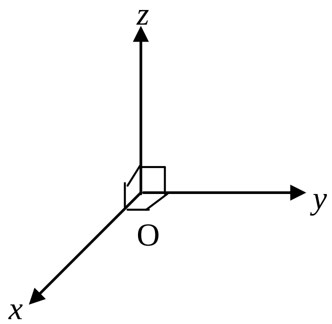


Figure 2.1

This consist of a fixed point O, the origin, and three mutually perpendicular axes, O*x* , O*y* and O*z*. The axes are placed in such a way that they form a right-handed set as shown in figure 2.1.

Each pair of coordinate axes determines a plane called a *coordinate plane*. These are referred to as the *xy-plane,* the *xz-plane* and the *yz-plane.*

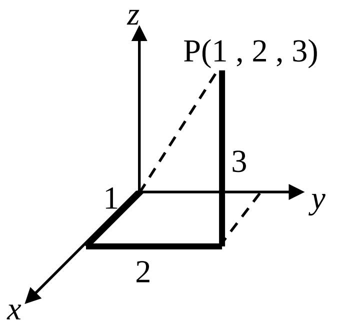


Figure 2.2

Any point *P* in space can be specified by an ordered triple of numbers (a , b , c) where a , b and c are the steps in the direction of x, y and z axes respectively, to *P*.

In figure 2.2, we have constructed the point $P(1 , 2 , 3)$.

We now take unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ in the direction of x, y and z axes respectively.

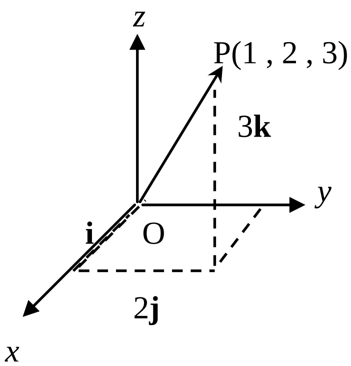


Figure 2.3

If $P(a , b , c)$ is any point in the space, then the position vector of P is

$$\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \quad \text{or} \quad < \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} >$$

In figure 2.3, the position vector of the point (1 , 2 , 3) is $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$. Conversely, the point whose position vector is $2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ has coordinates (2 , –4 , 1).

5.1 a) Types of Vectors

Position Vector

A vector that starts at the origin is called a position vector. So, the position vector of *B* is the vector that starts from the origin and ends at *B*; $\boldsymbol{b} = \overrightarrow{\mathrm{OB}}$. Similarly, position vector of *P* is $\boldsymbol{p} = \overrightarrow{\mathrm{OP}}$

Zero Vector

The zero vector , denoted by 0, has magnitude zero. Contrary to all the other vectors, it has no specific direction.

Unit vector

The unit vector of a vector **a** is a vector whose magnitude is 1 unit in the direction of ***a***.

The unit vector of **a** , $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$

Parallel vectors

If the vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are parallel, then they are scalar multiple of each other.

Thus, $\boldsymbol{u}_1 = \lambda \boldsymbol{u}_2$, $\lambda \in \Re$.

Perpendicular Vector

If vector ***a*** and ***b*** are perpendicular **,** hence the angle between ***a*** and ***b*** is 90° .

5.2 b) Addition and Scalar Multiplication of Vectors

Vector Arithmetic

Vectors in space apply the same rules of addition , subtraction, scalar multiplication and also the magnitude just as they are in the plane.

For any vectors $\mathbf{v_1} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v_2} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, and for any scalar *k*,

i) $|\mathbf{v_1}| = \sqrt{\boldsymbol{a}_1^{\,2} + \boldsymbol{b}_1^{\,2} + \boldsymbol{c}_1^{\,2}}$

ii) $\mathbf{v_1} + \mathbf{v_2} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$

$\mathbf{v_1} - \mathbf{v_2} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} + (c_1 - c_2)\mathbf{k}$

iii) $k\mathbf{v_1} = ka_1\mathbf{i} + kb_1\mathbf{j} + kc_1\mathbf{k}$

Example 1

Find $|2\mathbf{a} - \mathbf{b}|$ where $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

Example 2

Find a unit vector $\boldsymbol{u}$ in the direction of the vector from A(1 , 0 , 1) to B(3 , 2 , 0). Hence, find a vector 6 units long in that direction.

Example 3

If $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, express in terms of **i**, **j** and **k**
(a) $\mathbf{p} + \mathbf{q}$ (b) $\mathbf{q} - \mathbf{p}$

LECTURE 2 OF 7
TOPIC : 5.0 VECTORS
SUBTOPIC : 5.2 Scalar Product
LEARNING OUTCOMES : At the end of the lesson students should be able to:

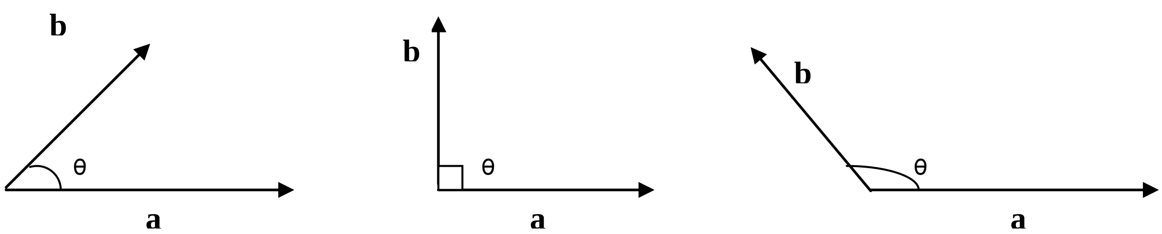
(a) find the scalar product
(b) use the properties of scalar product
(c) find the angle between two vectors
(d) find the direction cosines for a non-zero vector

5.2 a) <u>The Scalar Product (Dot Product)</u>

The scalar product of two vectors is $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is the operation which is written **a.b** and defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta \ ,$$

where θ $(0 \le \theta \le 180)$ is the angle between **a** and $\boldsymbol{b}$.



$\mathbf{a} \cdot \mathbf{b} > 0$ $\mathbf{a} \cdot \mathbf{b} = 0$ $\mathbf{a} \cdot \mathbf{b} < 0$

Definition

The scalar product between **a** and **b** is also defined as :-

a . b = $a_1b_1 + a_2b_2 + a_3b_3$

Example 1

Evaluate **a . b** if **a** = 3**i** – 7**k** , **b** = –2**j** + 3**k** .

Example 2

Evaluate a) (2**i** – **j**) **.** (3**i** + 4**k**)

b) (3**j** – 2**k**) **.** (**i** + 2**j** – 7**k**)

5.2 b) Properties of the scalar product

1. **a . a** = $|\mathbf{a}|^2$
2. **a . b** = **b . a**
3. **a .** (**b** + **c**) = **a . b** + **a . c**
4. (m**a**) **. b** = m (**a . b**) = **a .**(***m* b**)
5. **0.a** = 0
6. **a . b** = $|\mathbf{a}||\mathbf{b}|$ if and only if **a** parallel to **b**
   a . b = $-|\mathbf{a}||\mathbf{b}|$ if and only if **a** and **b** in opposite direction.
7. **a . b** = 0 if and only if **a** is perpendicular to **b** with **a** ≠ 0, **b** ≠ 0
8. For unit vectors **i, j** and **k** we have :-

   i . i = **j . j** = **k . k** = 1 and

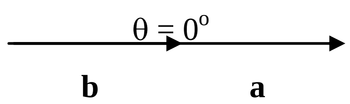
   i . j = **j . k** = **k . j** = 0

Proof

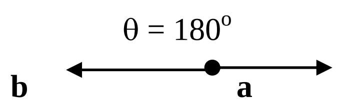
Proof for 1 to 4 are obvious

6. If vectors **a** and **a** parallel , the angle between **a** and **b** are 0^0 or 180^0.

So , **a . b** = $|\mathbf{a}||\mathbf{b}|\cos 0^o = |\mathbf{a}||\mathbf{b}|$

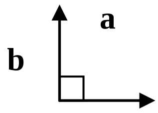


or, **a . b** = $|\mathbf{a}||\mathbf{b}|\cos 180^o = -|\mathbf{a}||\mathbf{b}|$



7. If vectors **a** and **b** are perpendicular the angle between **a** and **b** is 90^o so,

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90^{o} = 0$



8. It is known that **i**, **j**, and **k** are perpendicular each other and the angle of two parallel vectors is zero, so we have :-

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= a_1b_1(\mathbf{i} \cdot \mathbf{i}) + a_1b_2(\mathbf{i} \cdot \mathbf{j}) + a_1b_3(\mathbf{i} \cdot \mathbf{k}) + a_2b_1(\mathbf{j} \cdot \mathbf{i}) + a_2b_2(\mathbf{j} \cdot \mathbf{j}) + a_2b_3(\mathbf{j} \cdot \mathbf{k})$$
$$+ a_3b_1(\mathbf{k} \cdot \mathbf{i}) + a_3b_2(\mathbf{k} \cdot \mathbf{j}) + a_3b_3(\mathbf{k} \cdot \mathbf{k})$$

Since

$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$

thus

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Example 3

Simplify a) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$

b) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} - (\mathbf{a} + \mathbf{c}) \cdot \mathbf{b}$

Example 4

Given that $\mathbf{a} = 3\mathbf{i} + t\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = (1 - t)\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, find t if **a** is perpendicular to **b**.

5.2 c) The Angle Between Two Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors and θ is the angle between them. From the definition of **a . b** ,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\therefore \text{ The angle } \theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

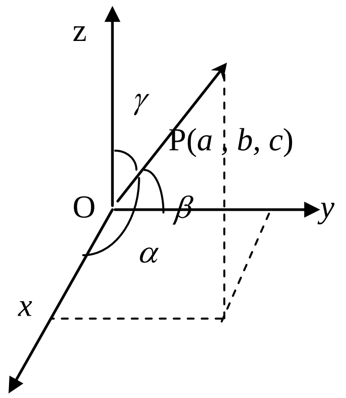
Example 5

If $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = 7$, find the angle between **a** and **b** .

Example 6

Find the interior angles of the triangle ABC whose vertices are A(1,3,5), B(-2,0,3) and C(3,1,-2).

5.2 d) <u>Direction Cosines for a Non-Zero Vector</u>



Consider the vector $\overrightarrow{OP}$ where P is the point (a , b , c).

Then $\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $|\overrightarrow{OP}| = \sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2}$

If $\overrightarrow{OP}$ makes angles of α , β and γ with the x , y and z-axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as :

Direction cosines;

$$\cos\alpha = \frac{a}{|\overrightarrow{OP}|} \quad , \quad \cos\beta = \frac{b}{|\overrightarrow{OP}|} , \quad \cos\gamma = \frac{c}{|\overrightarrow{OP}|}$$

Direction angles;

$$\alpha = \cos^{-1}\frac{a}{|\overrightarrow{OP}|} \quad , \quad \beta = \cos^{-1}\frac{b}{|\overrightarrow{OP}|} , \quad \gamma = \cos^{-1}\frac{c}{|\overrightarrow{OP}|}$$

where $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Notice that the unit vector $\hat{\overrightarrow{OP}} = \dfrac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \dfrac{\boldsymbol{a}}{|\overrightarrow{OP}|}\,\mathbf{i} + \dfrac{\boldsymbol{b}}{|\overrightarrow{OP}|}\,\mathbf{j} + \dfrac{\boldsymbol{c}}{|\overrightarrow{OP}|}\,\mathbf{k}$

$$\hat{\overrightarrow{OP}} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

Example 7

Find the direction cosine of the vector $\overrightarrow{OP}$ where P is the point (3 , -6 , 2)

Example 8

Find the direction cosines and direction angles of

a) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

b) $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

LECTURE 3 OF 7

TOPIC	**: 5.0 VECTORS**
SUBTOPIC	**: 5.4 The Vector Product**
LEARNING OUTCOMES	**:** At the end of the lesson students are able to: (a) find the vector product (b) use the properties of vector product

CONTENT

5.4 a) <u>The Vector Product</u>

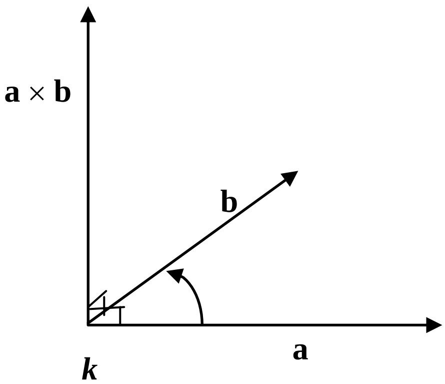
If θ is the angle between vector **a** and **b**, then

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\ \hat{\mathbf{u}}$$

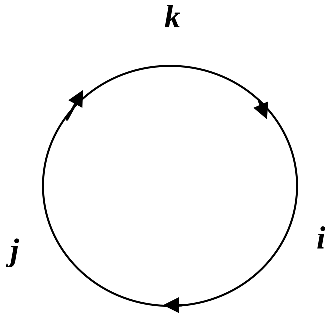
where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\mathbf{a} \times \mathbf{b}$

or $$\hat{\mathbf{u}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

To determine the direction of $\mathbf{a} \times \mathbf{b}$, use the right hand, where the fingers turn from **a** to **b** and the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.



Note



$\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

The vector product of $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is defined in terms of the expansion of the symbolic determinant ;

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Example 1

Given $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$.

Example 2

Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

a) Find $\mathbf{a} \times \mathbf{b}$
b) Prove that $\mathbf{a} \times \mathbf{b}$ is a vector which is perpendicular to the vector $\mathbf{a}$.

5.4 b) <u>Properties of vector product</u>

If **a** and **b** is a vector , *m* is a scalar , then

1. $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$
2. $(m\mathbf{a})$ x $\mathbf{b} = m(\mathbf{a} \times \mathrm{b}) = \mathbf{a} \times (m\,\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

4. (**a** + **b**) x **c** = **a** x **c** + **b** x c

5. **a.**(**b** x **c**) = (**a** x **b**) **. c**

6. **a** x (**b** x **c**) = (**a.c**)**b** – (**a.b**)**c**

7. **a** $\times$ **b** = **0** if **a** is parallel to ***b***

Example 3

Find all vectors of length $\sqrt{11}$ unit which are perpendicular to both **a** = **i** + 2**j** – **k** and **b** = **i** – 3**k**

Example 4

Given that a, b and r are three vectors whereas λ is a scalar such that $a \times r = b + \lambda a$ and $a \bullet r = 2$ .By using the result $a \times (b \times c) = (a \bullet c)b - (a \bullet b)c$, show that $r = \dfrac{2a - a \times b}{|a|^2}$.

Example 5

Given $a = 2i - j + 4k$ and $b = -6i + 3j - 12k$. By using the vector product show that a and b are parallel.

LECTURE 4 OF 7

TOPIC **: 5.0 VECTORS**

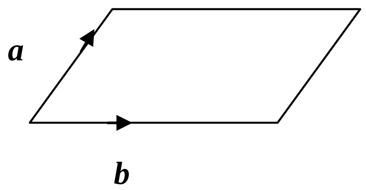
SUBTOPIC **: 5.4 The Vector Product**

LEARNING OUTCOMES **:** At the end of the lesson students are able to:

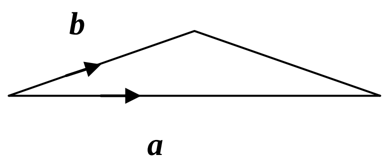
(c) find the area of parallelogram and a triangle.

CONTENT

5.4 c) <u>Area of Parallelogram</u>



Area of parallelogram = $|\mathbf{a} \times \mathbf{b}|$



Area of triangle = $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

Example 1

A plane contains points A(1,1,1), B(3,2,-1) and C(1,-4,2) and D. If A, B,C and D forms a parallelogram, find

a) the coordinates of D

b) the area of the parallelogram ABCD

Example 2

A plane contains points A(1,1,0) , B(3,-2,1) and C(5,7,2). Find

a) a vector normal to the plane

b) the area of triangle ABC

TOPIC **: 5.0 VECTORS**

SUBTOPIC **: 5.4 Application Of Vectors In Geometry**

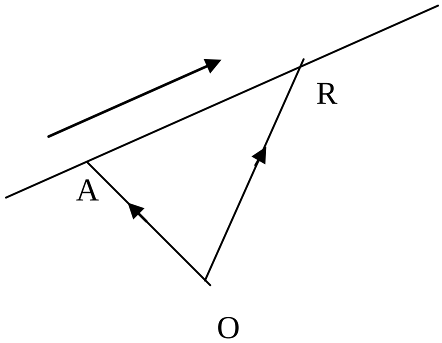
LEARNING OUTCOMES **:** At the end of the lesson students should be able to:

a) find equation of a straight line

CONTENT

5.4 a) <u>Lines In Space</u>

A **line** in space is a *straight line* which continues indefinitely in both directions and contains a continuous infinite set of points.



Suppose that R(x , y , z) is a point which is free to move on a line containing a fixed point A(x_1, y_1, z_1). If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a direction vector of the line, it is clear that a line consists precisely of those points for which the vector AR is parallel to **v**, that is

$$\overrightarrow{AR} = t\mathbf{v} \text{ for some scalar } t$$

$$\overrightarrow{OR} - \overrightarrow{OA} = t\mathbf{v}$$

$$\overrightarrow{OR} = \overrightarrow{OA} + t\mathbf{v}$$

or $\mathbf{r} = \mathbf{a} + t\mathbf{v}$ (1)

In terms of components, it can be written as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) + t\,(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

So that
$$\left.\begin{aligned} x &= x_1 + ta \\ y &= y_1 + tb \\ z &= z_1 + tc \end{aligned}\right\} \qquad (2)$$

Isolating t in each of these equations gives

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad (3)$$

So there are three ways of expressing the line in space :

(1) is the vector equation of the line,
(2) are the parametric equations of the line,
(3) are the Cartesian/ Symmetry equations of the line.

<u>Remarks</u>

If a straight line passes through A(x_1, y_1, z_1) and parallel to $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$,

- Its **vector equation** is $\mathbf{r} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) + t\,(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

- Its **parametric equations** are
$$x = x_1 + ta$$
$$y = y_1 + tb$$
$$z = z_1 + tc$$
t is called the *parameter* and can take any real value.

- Its **Cartesian / Symmetry equations** are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

Example 1

Find a vector equation of the line that contains points A(1,2,3) and B(-2,1,3)

Example 2

Find the vector equation, the parametric equations and the Cartesian equations of the line through (1, –2, 3) in the direction **4i + 5j – 6k**

Example 3

A line has Cartesian equations $\dfrac{x-1}{3} = \dfrac{y-2}{4} = \dfrac{z-3}{5}$. Find a vector equation for a parallel line passing through the point with position vector $5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and find the coordinates of the point on this line where $y = 0$.

Example 4

a) Find parametric equations for the line l passing through the points A (2 , 4 , –1) and B (5 , 0 , 7).

b) Where does the line intersect the xy-plane?

LECTURE 6 OF 7

TOPIC **: 5.0 VECTORS**

SUBTOPIC **: 5.4 Application Of Vectors In Geometry**

LEARNING OUTCOMES : At the end of the lesson students are able to:

(a) find the angle between two straight lines.

(b) find the equation of a plane.

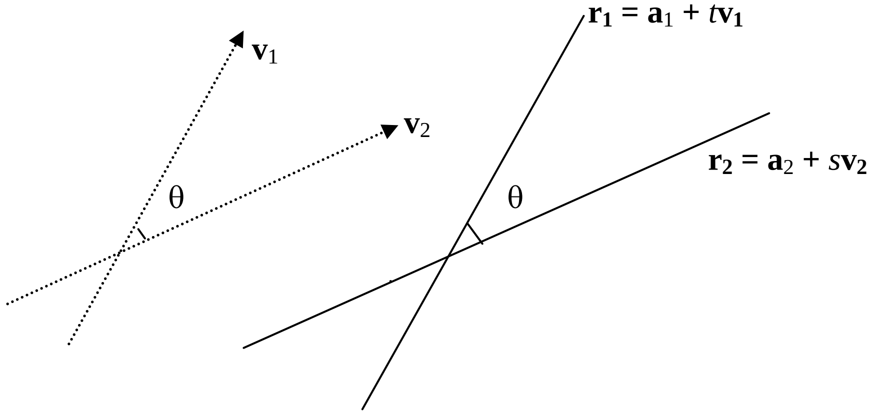
CONTENT

5.4 b) The Angle Between Two Straight Lines

Suppose two straight lines vector equation are

$\mathbf{r_1} = \mathbf{a}_1 + t\mathbf{v_1}$
$\mathbf{r_2} = \mathbf{a}_2 + s\mathbf{v_2}$

With *t*, *s* are any scalar and θ is angle between two straight lines.



If θ is angle between two straight lines, its also θ between $\mathbf{v_1}$ and $\mathbf{v_2}$. because of the lines are parallel.

$$\mathbf{v_1} \cdot \mathbf{v_2} = |\mathbf{v_1}||\mathbf{v_2}|\cos\theta$$

$$\cos\theta = \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{|\mathbf{v_1}||\mathbf{v_2}|}$$

Hence, the angle between two straight lines given by

$$\theta = \cos^{-1} \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{|\mathbf{v_1}||\mathbf{v_2}|}$$

Two straight lines are perpendicular if $\mathbf{v_1} \cdot \mathbf{v_2} = 0$ and the straight lines are parallel if $\mathbf{v_1} = k\mathbf{v_2}$ for *k* scalar.

Example 1

The vector form of two straight lines equation are given by

$\mathbf{p} = (2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$

and $\mathbf{q} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + s(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

with t and s scalar. Find the acute angle between the two straight lines.

Example 2

Find the acute angle between L_1 and L_2 for each of the following:

(a) $L_1 : r = 3i + 2j + t(i + j + k)$

$L_2 : \dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z+1}{4}$

(b) $L_1 : x = 1$, $\dfrac{y-1}{-2} = \dfrac{z-3}{4}$

L_2 is the y -axis

Example 3

Show that the line L_1 with vector equation $r = 2i - j + t(2i + j - k)$ is perpendicular to the line L_2 with Cartesian equations $\dfrac{x-1}{3} = \dfrac{y+1}{-2} = \dfrac{z}{4}$.

5.4 c) <u>Plane</u>

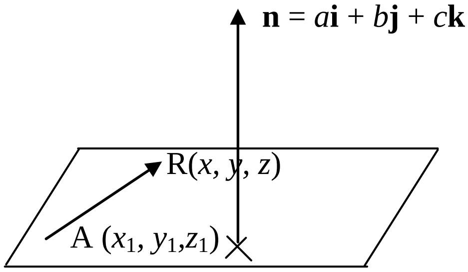
Planes equation in 3 dimension can be measure with this criteria :

1. A plane in space has normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and that it passes through the fixed point A (x_1 , y_1 ,z_1) or
2. A plane passes through 3 fixed point or
3. A plane has two vectors.

Suppose a plane in space has normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and that it passes through the fixed point A (x_1 , y_1 ,z_1).

R(x , y , z) moves any point on the plane.

Now $\overrightarrow{AR}$ is perpendicular to **n**.



$\therefore \quad \overrightarrow{AR} \cdot \mathbf{n} = 0$

$(\overrightarrow{OR} - \overrightarrow{OA}) \cdot \mathbf{n} = 0$

$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$

$\mathbf{r} \cdot \mathbf{n} - \mathbf{a} \cdot \mathbf{n} = 0$

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$\mathbf{r} \cdot \mathbf{n} = p$ where $p = \mathbf{a} \cdot \mathbf{n}$

Remarks: Equation of plane can be expressed in vector form or Cartesian/ Symmetry form

Example 1

Find the Cartesian equation of the plane with normal vector **i** + 2**j** + 3**k** and containing the point (-1 , 2 , 4).

Example 2

Find the equation of the plane passes through A(-1 , 2 , 0) , B(3 , 1 , 1) and C(1 , 0 ,3).

Example 3

Find the vector equation of the plane that contains (2,-1, 3) and is

a) parallel to the xy plane

b) parallel to the plane with equation $3x - y + z = 2$

Example 4

Show that the line L whose vector equation is $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ is parallel to the plane P whose vector equation $\mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$

Example 5

Find the symmetry equation of the plane passes through B(1,-1,3) and perpendicular to both the planes $x - y + 2z = 3$ and $2x + y - z = 3$.

LECTURE 7 OF 7

TOPIC : 5.0 VECTOR

SUBTOPIC : 5.4 Application of Vectors In Geometry

LEARNING OUTCOMES: At the end of the lesson students are able to:

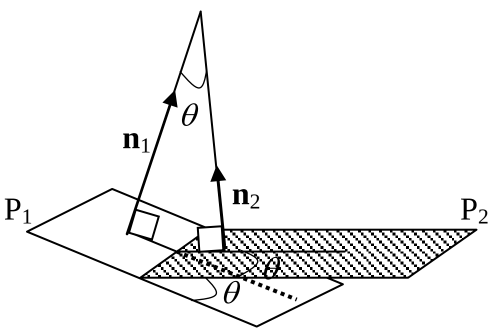
d) to find the angle between two planes

e) to find the angle between a line and a plane

f) to find the point of intersection between a line and a plane

CONTENT

5.4 d) The angle between two planes



Consider two planes P_1 and P_2 whose vector equations are

$\mathbf{r.n_1=d_1}$ and $\mathbf{r.n_2=d_2}$

The angle between P_1 and P_2 is equal to the angle between the normal to P_1 and P_2, i.e. the angle between n_1 and n_2.

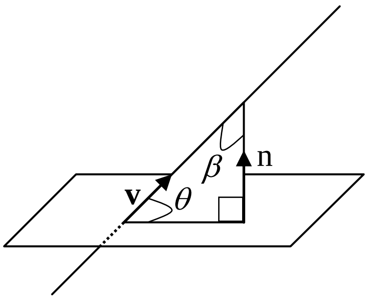
Therefore if θ is the angle between P_1 and P_2, $\cos\theta = \dfrac{\mathbf{n}_1 . \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$

Example 1
Find the angle between the two planes

$3x - 2y + 5z = 1$ and $4x + 2y - z = 4$

Example 2
Find the acute angle between the two planes $x + 5y + z = 0$ and $2x - y + 2z = 3$.

5.4 e) The angle between a line and a plane



Consider the line **r** = **a** + λ**v** and the plane **r**.**n** = d . The angle β between the line and the normal to the plane is given by

$$\cos\beta = \frac{\mathbf{v}.\mathbf{n}}{|\mathbf{v}||\mathbf{n}|}, \quad \beta = \cos^{-1}\frac{v.n}{|v||n|}$$

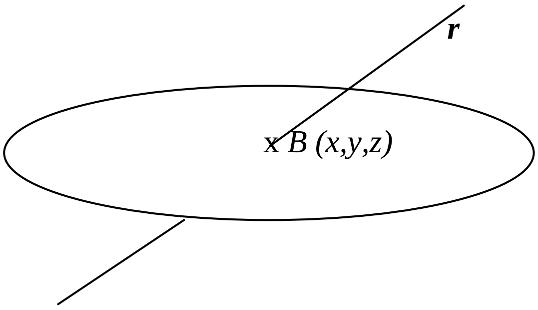
If θ is the angle between the line and the plane then $\theta = 90° - \beta$

Example 3

Find the angle between the straight line **r** = $\begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and the plane $x + y - 3z = 2$

5.4 f) Point of the intersection between a line and a plane

To find the point of the intersection between a line and a plane, we suppose to have a line through a plane. Let ***r = a + tv*** is a vector of the line and ***r.n*** = *d* is a plane. Where B the point of intersection between a line and a plane



From **r = a +** t**v,**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad x = x_1 + at, \quad y = y_1 + bt, \; z = z_1 + ct$$

From the plane equation, ***r.n*** = *d*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} p \\ q \\ r \end{pmatrix} = d \;, \; \therefore \; px + qy + rz = d \; \ldots\ldots\ldots \; (1)$$

Substitute $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$ into (1)

$$p(x_1 + at) + q(y_1 + bt) + r(z_1 + ct) = d$$

$\therefore t = \dfrac{d - (px_1 + qy_1 + rz_1)}{pa + qb + rc}$, then substitute *t* into $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$

Example 4

Find the vector equation of the line passing through the point (3 , 1 , 2) and perpendicular to the plane **r** . (2**i** – **j** + **k**) = 4 .Find also the point of intersection of this line and the plane.

Example 5

Find the point where the straight line **r** = (-4**i** + **j** + 9**k**) + λ (-2**i** + 4**k**) intersects the plane **r.**(2**i** + 2**j** – **k**) = 5.