LECTURE 1 OF 3

TOPIC : 3.0 NUMERICAL METHODS

SUBTOPIC : 3.1 Solutions of Non-Linear Equations

LEARNING OUTCOMES : At the end of this topic, students should be able to :

a) locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change.

SET INDUCTION :

Ask the students to solve the following equations :

1. Solve $x^2 - 4x - 5 = 0$

   Solution :

2. Solve $x^3 - 8 = 0$

   Solution :

3. Solve $x^3 - e^x = 0$

   Solution :

So, in this subtopic we are to going discuss numerical methods in finding the approximate numerical solutions of such equations.

CONTENT :

Introduction

Many equations cannot be solved exactly, but various methods of finding approximate numerical solutions exist.

The most commonly used methods have two main parts:

(a) finding an initial approximate value

(b) improving this value by an iterative process

Initial Values:

The initial value of the roots of $f(x) = 0$ can be located approximately by either a graphical or an algebraic method.

GRAPHICAL METHOD:

Either

(a) Plot (or sketch) the graph of $y = f(x)$.
The real roots are the points where the curve cuts the x axis.

Or

(b) Rewrite $f(x) = 0$ in the form $F(x) = G(x)$.
Plot (or sketch) $y = F(x)$ and $y = G(x)$. The real roots are at the points where these graphs intersect.

EXAMPLE 1

Find the approximate value of the equation $\ln x + x - 4 = 0$ by using the graphical method.

EXAMPLE 2

On the same axes, sketch $e^{-y} = x - 1$ and $y = x + 1$. Show that the equation $\ln(x-1) + x + 1 = 0$ has only one root .

SOLUTION

EXAMPLE 3

By sketching the graph of $y = x^3$ and $y = 3x - 1$ on the same coordinate axes, show that the equation $x^3 - 3x + 1 = 0$ has 3 real roots.

SOLUTION

Algebraic Method

- Find two values *a* and *b* such that *f*(*a*) and *f*(*b*) have different signs.
- At least one root must lie between a and b if *f*(*x*) is continuous.
- If more than one root is suspected between *a* and *b*, sketch *y* = *f*(*x*).

EXAMPLE 4

Show that the equation $x^3 + x - 4 = 0$ has a root between $x = 1.3$ and $x = 1.4$. Approximate the root to 2 decimal places.

SOLUTION

LECTURE 2 OF 3

TOPIC : 3.0 NUMERICAL METHODS

SUBTOPIC : 3.2 Newton-Raphson method

LEARNING OUTCOME : At the end of this topic, students should be able to :

a) find the root by the Newton-Raphson Method

CONTENT

Newton-Raphson Method

If x_1 is the first approximation to the root of the equation $f(x)=0$, then second, third,…. approximations are written as $x_2, x_3, x_4,........$ and are given by the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Repeat this process as required

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

EXAMPLE 1

Using the Newton-Raphson Method, find the solution of $f(x)=x+e^x=0$ near to $x=-0.5$ correct to 3 decimal places.

SOLUTION

EXAMPLE 2

Show that the equation $x^3 + x - 6 = 0$ has a root between 1 and 2.
Using the Newton-Raphson method with the starting point 1.6, determine an approximation to this root, giving your answer to 3 significant figures.

SOLUTION

EXAMPLE 3

Use Newton-Raphson method to find an approximate value of $\sqrt[3]{15}$. Give your answer correct to 3 decimal places.

SOLUTION

EXAMPLE 4

Sketch the graph of $y = e^x$ and y = 2 – x where x < 2, on the same axes. Get the first approximation, x_0 for the equation $e^x = 2 - x$ where $0 < x_0 < 1$. Hence, by using Newton-Raphson method, solve the equation of $e^{-x} = \frac{1}{2-x}$ for x < 2 to 3 decimal places.

SOLUTION

EXAMPLE 5

Show that the equation $2\sin x - x = 0$ has a root between $x = 1$ (radian) and $x = 2$ (radian). Find the root of equation by using Newton Raphson method, giving your answer correct to 2 decimal places.

SOLUTION

LECTURE 3 OF 3

SUBTOPIC : 3.3 The Trapezoidal Rule

LEARNING OUTCOMES:

At the end of this lesson, students should be able to:
a) use the trapezoidal rule to approximate definite integral.

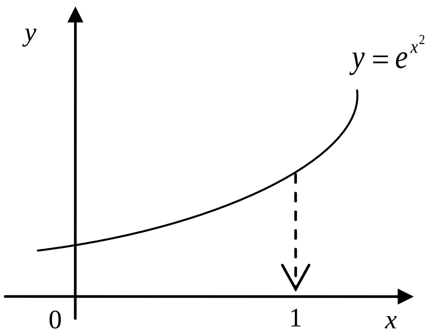
INDUCTION SET

Ask the students to integrate $\int_0^1 2x\,dx$ using their previous knowledge. But then, the teacher will explain to them that not every functions that can be integrated.

E.g: $\int_0^1 e^{x^2}\,dx$.

As some mathematical functions cannot be integrated, so we cannot find an exact value for them.

However, $\int_0^1 e^{x^2}\,dx$ is a definite integral and when we solve it we will get a number which represent the area between the curve $y = e^{x^2}$, the x-axis and the lines $x = 0$ and $x = 1$ as shown below.



In order to find $\int_0^1 e^{x^2}\,dx$, since e^{x^2} cannot be integrated, we have to evaluate the area using another method which is dividing the area into n trapezoids and then calculate the area of each trapezoid approximately. This method is known as the numerical method using the trapezium rule.

CONTENT

Introduction

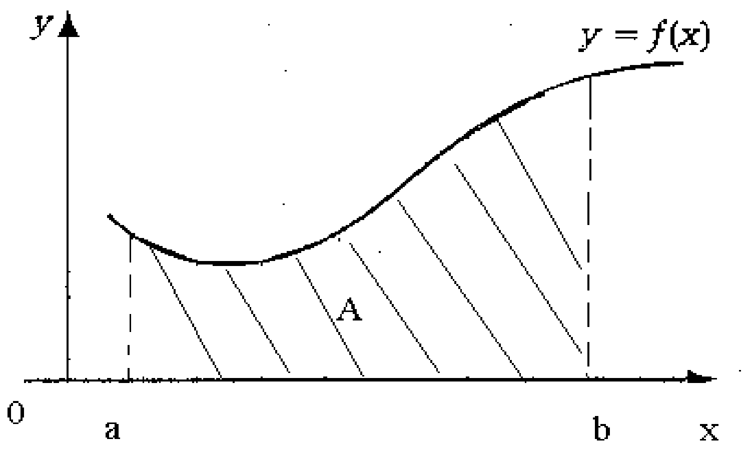
What happens when a mathematical function cannot be integrated? In these cases a numerical method can be used to find an approximate value for the integral. As the definite integral $\int_a^b f(x)dx.$ is a number which represents the area between $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. Therefore , even if $\int_a^b f(x)dx.$ cannot be found, the approximate value for $\int_a^b f(x)dx.$ can be found by evaluating the appropriate area using another method. The two common methods are the trapezium rule and Simpson's rule.

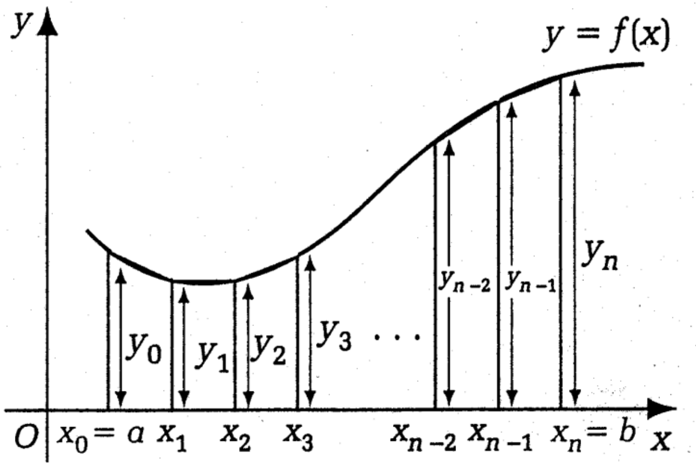
However, here we will discuss the trapezium rule as requested by the syllabus.

Trapezium Rule (Trapezoidal Rule)

The area A under the curve $y = f(x)$ bounded by $x = a$ and $x = b$ is divided into ***n*** strips with equal width ***h***. The area under the curve ≈ sum of areas of n trapezoid.

Note : ≈ approximately equal to





Let $y_0, y_1, y_2, \ldots, y_n$ be the values of the function $f(x)$ at ordinates $x_0, x_1, x_2, \ldots, x_n$ respectively. So,

Area of trapezium $= \frac{1}{2}(\text{width x sum 2 parallel sides})$

The area of the first trapezium is

$$A_1 = \frac{1}{2}h(y_\circ + y_1)$$

The areas of the second and the third trapezoids are then

$$A_2 = \frac{1}{2}h(y_1 + y_2) \qquad \text{and} \qquad A_3 = \frac{1}{2}h(y_2 + y_3)$$

Verify that if this process is continued then

$$A_{n-1} = \frac{1}{2}h(y_{n-2} + y_{n-1}) \qquad \text{and} \qquad A_n = \frac{1}{2}h(y_{n-1} + y_n)$$

Now add all of these separate areas together. The approximate value of A is given by

$$A \approx A_1 + A_2 + A_3 + \ldots + A_{n-1} + A_n$$

$$\approx \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \ldots + \frac{1}{2}h(y_{n-2} + y_{n-1}) + \frac{1}{2}h(y_{n-1} + y_n)$$

$$\approx \frac{1}{2}h(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n)$$

This method of approximating the area under a curve by n trapezoid of equal width h is called the trapezium rule, and the result can be summarized as :

$$\int_a^b f(x)\, dx \approx \frac{1}{2} h \left[y_0 + y_n + 2\left(y_1 + y_2 + \ldots + y_{n-1} \right) \right]$$

where $\boldsymbol{h} = \dfrac{b-a}{n}$ and $y_r = f(x_r)$

EXAMPLE 1

Evaluate $\int_2^6 \ln x \, dx$ by using trapezoidal rule with 4 strips (4 sub-intervals). Give your answer correct to 4 decimal places.

SOLUTION

(Notation: Working must have 5 or more decimal places.)

EXAMPLE 2

Approximate $\int_{3}^{5} \sqrt{1+x}\ dx$ by using trapezoidal rule with 4 strips (4 sub-intervals). Give your answer correct to 4 decimal places.

SOLUTION

(Notation: Working must have 5 or more decimal places.)

EXAMPLE 3

Use the trapezium rule, with five ordinates, to evaluate $\int_0^{0.8} e^{x^2}\,dx$. Correct your answer to four decimal places.

SOLUTION

EXAMPLE 4

Use trapezoidal rule with 5 ordinates to estimate $\int_0^{\pi} \sqrt{\sin x}\, dx$ correct to 4 decimal places.

SOLUTION

EXAMPLE 5

The region A in the first quadrant of the $x-y$ plane is bounded by the y-axis, the x- axis from 0 to $\frac{\pi}{2}$ and the curve $y = 2\left(\sqrt{1-\cos x}\right)\sin x$.

a) Find the area of A with integral method.

b) By using the trapezium rule with 3 intervals each of width $\frac{\pi}{6}$, to estimate the area of A, giving your answer correct to 3 decimal places.

c) Calculate the percentage error in your answer.

SOLUTION

EXERCISE

1. Estimate the value using trapezium rule correct to four decimal places of :

(a) $\int_0^{0.4} \sqrt{(1 - x^2)}\, dx$ with 5 ordinates **(Ans: 0.3887)**

(b) $\int_3^5 x \ln x \, dx$ with 4 sub-intervals **(Ans: 11.1849)**

(c) $\int_0^{\frac{\pi}{3}} \tan x \, dx$ with 4 strips **(Ans: 0.7098)**

2. Use the trapezium rule with only five equal sub-intervals to estimate the area of the region bounded by the curve $y = \sqrt{x(x+2)}$, the lines $x = 1$, $x = 4$ and the x-axis. Give your answer correct to four decimal places.

(Ans: 10.0237)