LECTURE   1 OF 3

TOPIC   :   **FIRST ORDER DIFFERENTIAL EQUATIONS**

SUBTOPIC   :   **2.1   Separable Variables.**

**LEARNING
OUTCOMES**   :   At the end of the lesson, students should be able to

a) distinguish between general and particular solutions
b) solve separable differential equations

CONTENT

Introduction

A simple type of differential equation is $\dfrac{dy}{dx} = x^2 - 3$. We integrate it to produce the required solution. Differential equation can be regarded as a mathematical model for many practical situations such as population growth, radioactive decay, chemical mixture, temperature cooling, velocity and acceleration problem and *etc*.

Definition 1

A differential equation (DE) is one which relates an independent or dependent variables with one or more derivatives.

Examples of differential equations:

a) $dy = (\sin x)dx$

b) $\dfrac{ds}{dt} = 2t^3 + 5$

c) $\left(\dfrac{dy}{dx}\right)^2 + y^2 = \sin x$

d) $3\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} - 7 = 0$

e) $y'' + 2y' - 5y = 4$

Definition 2

Order is the highest derivative in a differential equation.

Degree is the highest power of the highest derivative which occurs in a differential equation.

Example

Determine the order and degree of differential equations below.

Differential Equations	Order	Degree
1. $y^5 \dfrac{dy}{dx} = x^2 + 5$	1	1
2. $7\dfrac{d^2 y}{dx^2} = \dfrac{dy}{dx} - 3$	2	1
3. $\left(\dfrac{dy}{dx}\right)^6 = \left(\dfrac{d^3 y}{dx^3}\right)^4 + 1$		
4. $\left(\dfrac{dy}{dx}\right)^3 - 3x = \ln x$		
5. $\left(y''\right)^4 + 2\left(y'\right)^7 - 5y = 3$		
6. $\left(\dfrac{dy}{dx}\right)^5 - 2x = 3\sin x - \sin y$		

In this chapter we only deal with first order, first degree differential equations.

A differential equation has **general solution** and **particular solution**.

Definition 3

General solution –The general solution of a differential equation contains an arbitrary constant c.

Particular solution - The particular solution of a differential equation contains a specified initial value and containing no constant.
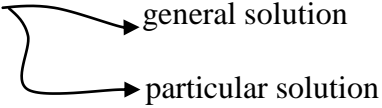
Differential Equations with Separable Variables

Definition :

Differential equations with separable variables –Differential equations in which the variables can be algebraically separated.

A differential equation is separable if it can be written in the form :

$$\frac{dy}{dx} = P(x)Q(y) \quad \text{or} \quad \frac{dy}{dx} = \frac{P(x)}{Q(y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q(y)}{P(x)}$$

To solve the equations:

    (i)   Separate the variables x and y
            (x by the side of dx and y by the side of dy)

    (ii)   Integrate both sides independently.

    (iii)   2 types of solution   general solution

                                        particular solution

Situation 1

Given $\dfrac{dy}{dx} = 2x$

Situation 2

Given $\dfrac{dy}{dx} = 2x$, when x = 1 , y = 2

Example 1

Solve $y^2 dy + x^3 dx = 0$.

Solution

Example 2

Solve $\dfrac{\sin x}{1-y}\dfrac{dy}{dx} = \cos x.$

Solution

Example 3

Solve $e^{-x}\dfrac{dy}{dx} = (1-y)^2$.

Solution

Example 4

Solve the differential equation $x\sqrt{x^2+1} - ye^y\dfrac{dy}{dx} = 0$.

Solution

Example 5

Solve the differential equation $\dfrac{dy}{dx} = x(y-2)$ when $y(0) = 5$.

Solution

Example 6

Find the particular solution of differential equation

$$\left(1 + \sin^2 x\right)\frac{dy}{dx} = e^{-2y}\sin 2x, \quad y(0) = 1.$$

Solution

Example 7

Find y in terms of x given that $2\dfrac{dy}{dx} + y^2 = 4$, hence find the particular solution if given $y(\ln 2) = 0$.

Solution

Exercise

Find the solutions of the following differential equations:

1. $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1+3y)}$

2. $\dfrac{dy}{dx} = \dfrac{\sec^2 y}{1+x}$

3. $\dfrac{dy}{dt} = \dfrac{6t^5 - 2t + 1}{\cos y + e^y}$

4. $x\dfrac{dy}{dx} = xy + y$, $y(3) = 2$

5. $xy\,dx + (1+x^2)\,dy = 0$, $y(1) = 2$

6. $2(x^2+1)\dfrac{dy}{dx} = x(4-y^2)$, $y(0) = 1$

Answers

1. $\ln|y| + 3y = 2\ln|x| - 3x + c$

2. $\sin 2y + 2y = 4\ln|1+x| + A$

3. $\sin y + e^y = t^6 - t^2 + t + c$

4. $y = \dfrac{2}{3}e^{-3}xe^x$

5. $y^2 = \dfrac{8}{1+x^2}$

6. $y = \dfrac{6x^2+4}{3x^2+4}$

LECTURE   2 OF 3

TOPIC   :   **FIRST ORDER DIFFERENTIAL EQUATIONS**

SUBTOPIC   :   **2.2   First Order Linear Differential Equation**

LEARNING
OUTCOMES   :   At the end of the lesson, students should be able to

a) solve first order linear differential equations by means of
integrating factor.

CONTENT

Linear First-Order Differential Equations

A linear first-order differential equation is one of the form $\dfrac{dy}{dx} + P(x)y = Q(x)$.

The solution must be on an interval where both $P(x)$ and $Q(x)$ are continuous.
The usual solution to the differential equation is to change it to an exact equation by means of

an **integrating factor**. This integrating factor is $V(x) = e^{\int P(x)dx}$.

By multiplying both sides of the equation by the integrating factor $V(x)$, you change the left-hand side of the equation into the derivative of the product $V(x)y$

$$V(x)\frac{dy}{dx} + P(x)V(x)y = V(x)Q(x)$$

Once we have chosen $V(x)$, (we shall do so in a moment) and carried out the multiplication ,

$$V(x)\frac{dy}{dx} + P(x)V(x)y = V(x)Q(x)$$

$$\frac{d}{dx}[V(x)y] = V(x)Q(x) \qquad \text{Choice of } V(x)$$

we can solve the above equation by integrating both sides .

$$\int d[V(x)y] = \int V(x)Q(x)\, dx$$

Example 1

Solve $\dfrac{dy}{dx} - \dfrac{y}{(x+1)} = x$

Solution

Example 2

Solve $r\dfrac{dt}{dr} - 2t + r = 0$.

Solution

Example 3

Solve $\dfrac{dy}{dt} + \dfrac{2}{t}y = \dfrac{\cos t}{t^2}$, $y(\pi) = 0$

Solution

Example 4

Solve $e^t \dfrac{dr}{dt} + 2e^t r = 3$, $r(0) = 2$

Solution

Example 5

Solve $(y+1)\dfrac{dx}{dy} + x = 2y - 1$, given that $x = 1, y = 1$

Solution

Exercises

1. Solve :

a) $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1+3y)}$

b) $\dfrac{dy}{dx} = \dfrac{\sec^2 y}{1+x}$

c) $\dfrac{dy}{dt} = \dfrac{6t^5 - 2t + 1}{\cos y + e^y}$

2. Find the particular solutions of the following differential equations:

a) $x\dfrac{dy}{dx} = xy + y$, $y(3) = 2$

b) $xydx + (1+x^2)dy = 0$, $y(1) = 2$

c) $2(x^2 + 1)\dfrac{dy}{dx} = x(4 - y^2)$, $y(0) = 1$

Answers

1)   a) $\ln|y| + 3y = 2\ln|x| - 3x + c$

        b) $\sin 2y + 2y = 4\ln|1+x| + A$

        c) $\sin y + e^y = t^6 - t^2 + t + c$

2)   a) $y = \dfrac{2}{3} e^{-3} xe^x$ b) $y = \dfrac{8}{1+x^2}$ c) $y = \dfrac{6x^2 + 4}{3x^2 + 4}$

LECTURE   3 OF 3

TOPIC   :   **FIRST ORDER DIFFERENTIAL EQUATIONS**

SUBTOPIC   :   **2.3   Applications of Differential Equations**

**LEARNING
OUTCOMES**   :   At the end of the lesson, students should be able to

a)   Solve problems that can be modeled by differential equations.

CONTENT

Applications of Differential Equations

Now we will apply the methods in this section to the solution of some practical situations.

A) Population growth model

The simplest growth model has a constant relative growth rate. If we denote the population we are considering by $y(t)$, then the rate of change of the population is $\dfrac{dy}{dt}$. To say that the rate of change is proportional to the population is just saying that there is a constant of proportionality k such that

$$\frac{dy}{dt} = ky$$

Since k is constant, this can be immediately separated and integrated to yield

$$\frac{dy}{y} = kdt$$

$$\int \frac{dy}{y} = \int kdt$$

$$\ln y = kt + c$$

$$y = e^{kt+c}$$

$$y = e^{kt} \times e^{c}$$

$$y = Ae^{kt}$$

Example 1

In a particular bacteria culture, the rate of increase of bacteria is proportional to the number of bacteria, N, present at time t hours after the experiment.

Given that the number of bacteria at the beginning is 10^6, and after 1 hour is 10^9. Find
   a) The number of bacteria after 5 hours
   b) The time taken for the number of bacteria to be 3 times the original.

Solution

B) Radioactive decay models

Radioactive decay models, on the other hand, are very accurate over long periods of time. They are the primary method for determining age of prehistoric fossils and ancient artifacts. If we denote the decay we are considering by $C(t)$, then the decreasing rate of the decay is $\dfrac{dC}{dt}$. To say that the decreasing rate is proportional to the decay is just saying that there is a constant of proportionality k such that

$$\frac{dC}{dt} = -kC$$

Since k is constant, this can be immediately separated and integrated to yield

$$\frac{dC}{C} = -kdt$$
$$\int \frac{dC}{C} = \int -kdt$$
$$\ln C = -kt + c$$
$$C = e^{-kt+c}$$
$$C = e^{-kt} \times e^{c}$$
$$C = Ae^{-kt}$$

Example 2

Radium decomposes at a rate which is proportional to the amount present at any time. If 10% decomposes in 200 years, what percentage of the original amount of radium will remain after 1000 years?

Solution

C) Newton's Law of Cooling

When an object has a temperature greater than the ambient temperature, it cools according to Newton's Law of cooling which states that the rate of cooling is proportional to the difference in the temperatures, that is: $\dfrac{d\theta}{dt} = -k(\theta - a)$, where $\theta(t)$ is the temperature of the object at any time t and a is the ambient temperature. The solution to this separable differential equation is:

$$d\theta = -k(\theta - a)dt$$

$$\frac{d\theta}{\theta - a} = -kdt$$

$$\int \frac{d\theta}{\theta - a} = \int -kdt$$

$$\ln(\theta - a) = -kt + c$$

$$e^{-kt+c} = \theta - a$$

$$\theta = e^{-kt+c} + a$$

$$\theta = Ae^{-kt} + a$$

Example 3

A body temperature is 180^0C is cooled by immersing in a liquid at 60^0C. In one minute, the temperature of the body has fallen to 120^0C. How long will it take for the temperature of the body to fall to 90^0C

Solution

D) Electric Circuits

An RL circuit is one with a constant resistant, R, and a constant inductance, L. The electromotive force (EMF), a resistor and an inductor connected in series.

The EMF source which is usually a battery or generator supplies voltage that causes a current flow in the circuit.

According to Kirchhoff's second law, if the close circuit, time t = 0, then the applied electromotive force is equal to the sum of the voltage drops in the rest of the circuit. It can be shown that this implies that the current I(t) that flows in the circuit at time t must satisfy the first-order linear differential equation

$$L\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$P(t) = \dfrac{R}{L}$ (a constant) and $Q(t) = \dfrac{E}{L}$ (a constant)

The simplest model of the amount of current I in a simple electrical RL circuit is given by a linear first-order differential equation,

$$\frac{dI}{dt} + P(t)I = Q(t)$$

Where I = amount of current and t = time.

The usual solution to the differential equation is to change it to an exact equation by means of an *integrating factor*. This integrating factor is $V(t)$.

Example 4

The basic equation governing the amount of current I in a simple RL circuit is given by $\dfrac{dI}{dt} + 50I = 5$. When $t = 0, I = 0$, find the current at any time t.

Solution

Exercises

1. If P is the number of cells in a certain bacteria population, then it is reasonable to expect that the rate of growth of the bacteria population $\dfrac{dP}{dt}$ varies directly with P. A bacteria culture is started with 1000 bacteria, and after 2 hours it is estimated that there are now 8000 bacteria. Find the bacteria population to become 15000.

Ans: 2.6 hours.

2. The rate of decay of a radioactive substance varies with the mass of the substance remaining. Write a differential equation that represents this relationship. If the mass of the substance remaining after 12 years is half of the original mass, what is the percentage of the substance remaining after 6 years?

Ans: 70.7%

3. Under certain conditions, the rate of cooling of an abject varies with the difference between the temperature of the object and the room temperature. Given that the temperature of object at any time t is θ and the constant room temperature is θ_0, write down a differential equation to describe the rate of cooling of the object. Given that the room temperature is $20^0 C$.

   a) Find a general solution of this differential equation
   b) If it takes 12 minutes for the object to cool from $100^0 C$ to $50^0 C$, find the time taken for the object to cool from $50^0 C$ to $25^0 C$

Ans: 21.92 minutes

4. The basic equation governing the amount of current I in a simple RL circuit is given by $\dfrac{dI}{dt} + 10I = 2$. When $t = 0$, $I = 0$, find the current at $t = 5$ second.

Ans: $I = \dfrac{1}{5}\left[1 - e^{-10t}\right]$, $I = \dfrac{1}{5}$