LECTURE 1 OF 9

TOPIC	:	**1.0 INTEGRATION**
SUBTOPIC	:	**1.1 INTEGRATION OF FUNCTIONS**
LEARNING OUTCOMES	:	**At the end of the lesson, students should be able to:**
		(a) relate integration and differentiation
		(b) use the basic rules of integration

Integration as Anti-differentiation

Integration is the reverse process of differentiation. In differentiation, we start with an expression and then proceed to find its derivative. In integration, we start with the derivative and then work backward to find the function from which it is derived.

The process of finding anti-derivatives is called *anti-differentiation* or *integration*. If $\frac{d}{dx}\left[F(x)\right]= f(x)$, then the functions of the form $F(x)+C$ are the anti-derivatives of $f(x)$. We denote this by writing $\int f(x)\ dx = F(x) + C$.

"$\int$ " is called an integral sign and C is the constant of integration.

Basic Rules of Integration.

(i) $\int k\ dx = kx + \mathrm{c}$

(ii) $\int x^n\ dx = \dfrac{x^{n+1}}{n+1} + C\ ,\ n \neq -1$

(iii) $\int k\ f(x)\, dx = k \int f(x)\, dx$

(iv) $\int [\ f(x) \pm g(x)\ ]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

(v) $\int (ax+b)^n\ dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c\ ,\ n \neq -1$

Example 1:

Evaluate the following integrals:

a) $\int 3\, dx$

b) $\int x^6\, dx$

c) $\int \frac{1}{\sqrt{x}}\, dx$

Solution:

Example 2:

Evaluate the following integrals:

a) $\int -2\sqrt{x}\, dx$

b) $\int \frac{4x+1}{2x^3}\, dx$

Solution:

Example 3:

Evaluate the following integrals

a) $\int \left(6x^3 - 2x^2\right) dx$

b) $\int \left(x^2 + 5\right)^2 \, dx$

c) $\int (2x-1)(3x+2) \, dx$

d) $\int (5x+7)^9 \, dx$

Solution:

Exercises 1

Find the following integrals.

1. $\int 7z^{\frac{3}{4}}\ dz$
2. $\int \dfrac{5x+4}{x^3}\ dx$
3. $\int x^2(x-2)\ dx$
4. $\int \left(2x-\dfrac{1}{x}\right)^2\ dx$
5. $\int \left(4x^2-5x^3\right) dx$
6. $\int (3-9x)^5\ dx$

Answer:

1. $4z^{\frac{7}{4}}+c$
2. $-\dfrac{5}{x}-\dfrac{2}{x^2}+c$
3. $\dfrac{x^4}{4}-\dfrac{2x^3}{3}+c$
4. $\dfrac{4x^3}{3}-4x-\dfrac{1}{x}+c$
5. $\dfrac{4x^3}{3}-\dfrac{5x^4}{4}+c$
6. $-\dfrac{1}{54}(3-9x)^6+c$

LECTURE 2 OF 9

TOPIC : 1.0 INTEGRATION

SUBTOPIC : 1.1 INTEGRATION OF FUNCTIONS

LEARNING OUTCOMES : At the end of the lesson, students should be able to:

(a) determine integration of $\frac{1}{x}$, e^x and a^x

(b) determine the integral of the forms:

(i) $\int \frac{f'(x)}{f(x)}\,dx$

(ii) $\int f'(x)e^{f(x)}\,dx$

(iii) $\int f'(x)[f(x)]^n\,dx$

(iv) $\int a^{px+q}\,dx$

The Integral of $\frac{1}{x}$ (special case of $\int x^n dx$, when $n=-1$)

Remembered that $\frac{d}{dx}(\ln x)=\frac{1}{x}$. Therefore, reversing the process gives $\int \frac{1}{x}\,dx = \ln|x| + C$

Example 1:

Find the following integrals.

a) $\int \frac{2}{x}\,dx$

b) $\int \frac{1}{3x}\,dx$

Solution:

The integral of $\frac{f'(x)}{f(x)}$

In general, $\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)} \qquad \Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$

Example 2:

Find the following integrals.

a) $\int \frac{2x}{x^2-1}\,dx$ b) $\int \frac{10}{2x-1}\,dx$ c) $\int \frac{3x^2+x}{2x^3+x^2}\,dx$

Solution:

The integral of $f'(x)e^{f(x)}$

$$\frac{d}{dx}\left(e^{x}\right)=e^{x} \quad \Leftrightarrow \quad \int e^{x}\,dx = e^{x}+c$$

$$\frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)} \quad \Leftrightarrow \quad \int f'(x)e^{f(x)}\,dx = e^{f(x)}+c$$

Example 3:

Find the following integrals.

a) $\int 3e^{3x}\ dx$ b) $\int \left(3x^{2}+1\right)e^{x^{3}+x-2}\ dx$ c) $\int e^{5x}\ dx$

Solution:

The integral of $f'(x)[f(x)]^n$

If $f(x)$ is a function of x and $\dfrac{d}{dx}[f(x)] = f'(x)$,

$$\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}\right\} = f'(x)[f(x)]^n .$$

Reversing the process,

$$\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

Example 4:

Find the following integrals.

a) $\int 2x(x^2 - 3)^4 \, dx$

b) $\int \dfrac{\ln x}{x} \, dx$

c) $\int x^2 \sqrt{x^3 - 5} \, dx$

Solution:

The Integral of a^{px+q}

NOTE: $\dfrac{d}{dx}(a^x) = a^x \ln a \quad \Leftrightarrow \quad \int a^x \, dx = \dfrac{a^x}{\ln a} + c$

Example 5:

Find $\int 2^x \; dx$

Solution:

NOTE: $\frac{d}{dx}(a^{px+q}) = a^{px+q} p \ln a \quad \Leftrightarrow \quad \int a^{px+q}\, dx = \frac{a^{px+q}}{p \ln a} + c$

Example 6:

Find $\int 3^{2x-1}\, dx$

Solution:

Exercises 2

Questions :		Answer:
1.	$\int \left(\frac{3}{x^2} - \frac{7}{8x} \right) dx$	$\frac{-3}{x} - \frac{7}{8} \ln\lvert x\rvert + c$
2.	$\int \frac{2-3x}{3+4x-3x^2}\, dx$	$\frac{1}{2} \ln\lvert 3+4x-3x^2\rvert + c$
3.	$\int x\sqrt{x^2-1}\, dx$	$\frac{1}{3}\left(x^2-1\right)^{\frac{3}{2}} + c$
4.	$\int \frac{x^2}{\sqrt{x^3+1}}\, dx$	$\frac{2}{3}\left(x^3+1\right)^{\frac{1}{2}} + c$
5.	$\int \frac{2-e^{4x}}{e^x}\, dx$	$-2e^{-x} - \frac{1}{3}e^{3x} + c$
6.	$\int \left(2^{x+1} + 2^{-x}\right) dx$	$\frac{2^{x+1}}{\ln 2} - \frac{2^{-x}}{\ln 2} + c$

LECTURE 3 OF 9
TOPIC : 1.0 INTEGRATION
SUBTOPIC : 1.2 INTEGRATION OF TRIGONOMETRIC FUNCTIONS
LEARNING OUTCOMES : At the end of the lesson, students should be able to integrate trigonometric functions in the form $\sin ax$, $\cos ax$, $\sec^2 ax$, $\sin^2 ax$ and $\cos^2 ax$.

Integration of Trigonometric Functions

$$\frac{d}{dx}(\sin ax) = a\cos ax \quad \Leftrightarrow \quad \int \cos ax \, dx = \frac{1}{a}\sin ax + c$$

$$\frac{d}{dx}(\cos ax) = -a\sin ax \quad \Leftrightarrow \quad \int \sin ax \, dx = -\frac{1}{a}\cos ax + c$$

$$\frac{d}{dx}(\tan ax) = a\sec^2 ax \quad \Leftrightarrow \quad \int \sec^2 ax \, dx = \frac{1}{a}\tan ax + c$$

Example 1:

Find the following integrals.

a) $\int \sin 4x \, dx$

b) $\int 2\cos\left(\frac{3x}{4}\right) dx$

c) $\int \sec^2\left(\frac{x}{2}\right) dx$

d) $\int (3\sin 2x + \cos 6x) \, dx$

e) $\int (\cos 7x - 5\sec^2 3x) \, dx$

f) $\int \left(\frac{3}{\cos^2 2x} - \cos 4x\right) dx$

Solution:

Using Double Angle Formulae

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\quad\quad = 2\cos^2 x - 1$

$\quad\quad = 1 - 2\sin^2 x$

$\Rightarrow$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$

Example 2:

Find the following integrals.

a) $\int \sin^2 x \, dx$ b) $\int 2\cos^2 x \, dx$ c) $\int 3\sin^2\left(\dfrac{x}{2}\right) dx$ d) $\int 4\sin x\cos x \, dx$

Solution:

Using Compound Angle Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Example 3:

Find the following integrals.

a) $\int (\sin x \cos 2x + \cos x \sin 2x)\, dx$ b) $\int (\cos 2x \cos 3x - \sin 2x \sin 3x)\, dx$

Solution:

Exercises 3

Questions :		**Answer:**
1.	$\int \left(\sec^2 2x - \cos 3x\right) dx$	$\dfrac{\tan 2x}{2} - \dfrac{\sin 3x}{3} + c$
2.	$\int \dfrac{9}{2\cos ec 3x}\, dx$	$-\dfrac{3}{2}\cos 3x + c$
3.	$\int \left[\dfrac{1}{3\sec^2 5x} + \sin^2\left(\dfrac{x}{2}\right)\right] dx$	$\dfrac{2}{3}x + \dfrac{1}{60}\sin 10x - \dfrac{1}{2}\sin x + c$
4.	$\int \sin\left(\dfrac{2x}{3}\right)\cos\left(\dfrac{2x}{3}\right) dx$	$-\dfrac{3}{8}\cos\left(\dfrac{4x}{3}\right) + c$
5.	$\int \left(\sin x \sin 2x - \cos x \cos 2x\right) dx$	$-\dfrac{1}{3}\sin 3x + c$

LECTURE 4 OF 9
TOPIC : 1.0 INTEGRATION
SUBTOPIC : 1.3 TECHNIQUES OF INTEGRATION
LEARNING OUTCOMES : At the end of the lesson, students should be able to use substitution method to find integrals.

Integration by Substitution

Integration by substitution is used to compute indefinite integrals of the form

$$\int f[g(x)]g'(x)dx \quad \ldots\ldots\ldots \ (1)$$

To compute (1) we make a change of variable, or substitution.

We let $u = g(x)$ ………….. (2)

then $\dfrac{du}{dx} = g'(x)$ or $du = g'(x)dx$ ………….(3)

Making these substitutions (2) and (3) in (1), we obtain the integral

$$\int f(g(x))g'(x)dx = \int f(u)du$$

This method is illustrated in the following examples.

Example 1

Find $\int 3(2x+1)^{\frac{2}{3}}\ dx$

Solution:

Example 2

Find $\int (2x+3)\sqrt{x+1}dx$

Solution:

Note: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b)+C$

Example 3

Find $\int \frac{1}{5-3x}\,dx$

Solution:

Example 4

Find $\int \frac{3e^{1/t}}{t^2}\,dt$

Solution:

Example 5

Find $\int 5^{2x-3}\ dx$

Solution:

Example 6

Find $\int \frac{(\ln 2x)^3}{x} dx$

Solution :

Example 7

Prove that $\int \frac{2x^2}{x-1}dx = 2\int \frac{(u+1)^2}{u}du$. Hence, find $\int \frac{2x^2}{x-1}dx$.

Solution:

Example 8

Find $\int 3x^2 \sin x^3 dx$

Solution:

Example 9

Find $\int \cot x dx$

Solution:

Example 10

Find $\int \cos^3 x \sin x \ dx$

Solution:

Exercise 4

Find the following integrals:

a) $\int \frac{t}{(3t^2+1)^4}\,dt$ b) $\int \frac{1}{2x\ln\sqrt{x}}\,dx$ c) $\int 2x^3 e^{x^4}\,dx$ d) $\int \cos 2x \sin^4 2x\,dx$

Answers:

a) $-\frac{1}{18(3t^2+1)^3}+c$ b) $\ln\left|\ln\sqrt{x}\right|+c$ c) $\frac{1}{2}e^{x^4}+c$ d) $\frac{1}{10}\sin^5 2x+c$

LECTURE 5 OF 9
TOPIC : 1.0 INTEGRATION
SUBTOPIC : 1.3 TECHNIQUES OF INTEGRATION
LEARNING OUTCOMES : At the end of the lesson, students should be able to perform integration by parts.

Integration by Parts

Suppose u and v are differentiable functions of x. By the product rule of differentiation

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{............ (1)}$$

Integrating both sides of equation (1) with respect to x,

$$\int \frac{d(uv)}{dx}\,dx = \int \left(u\frac{dv}{dx} + v\frac{du}{dx} \right) dx$$

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Thus, $$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

This technique is called ***integration by parts***. The idea is to choose u and $\frac{dv}{dx}$, so that $\int v\frac{du}{dx}\,dx$ is simpler than the $\int u\frac{dv}{dx}\,dx$.

Example 1

Find $\int x\sqrt{x+1}\,dx$

Solution*:*

Example 2

Find $\int xe^{x}\,dx$

Solution :

Example 3

Find $\int 2xe^{-3x+1}\,dx$

Solution:

Example 4

Find $\int \ln x \, dx$

Solution :

Example 5

Find $\int x \ln x \, dx$

Solution

Example 6

Find $\int x^2 e^x \, dx$

Solution

Example 7

Find $\int x^2 \sin 2x \, dx$

Solution :

Exercises 5

Find the following integrals:

a) $\int \frac{\ln x}{x^3} dx$

b) $\int xe^{1+2x} dx$

c) $\int x\cos 3x dx$

d) $\int t^3 e^{-3t} dt$

Answers:

a) $-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$

b) $\frac{xe^{1+2x}}{2} - \frac{1}{4}e^{1+2x} + c$

c) $\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c$

d) $-\frac{1}{3}t^3e^{-3t} - \frac{1}{3}t^2e^{-3t} - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t} + c$

LECTURE 6 OF 9
TOPIC : 1.0 INTEGRATION
SUBTOPIC : 1.3 TECHNIQUE OF INTEGRATION
LEARNING OUTCOMES : At the end of the lesson, student should be able to evaluate the integral of a rational function by means of decomposition into partial fractions.

Rational Functions

A rational function is a fraction where both the numerator and the denominator are polynomials

Example: $\frac{x}{1+x^2}$, $\frac{x-1}{x+1}$, $\frac{3}{(x+1)(x-2)}$, $\frac{2x-1}{(x+2)(3x-4)}$, $\frac{x^2}{x-1}$

If the degree of the numerator is ***less*** than the degree of the denominator, the fraction is called ***proper fraction*** and if the degree of the numerator is ***greater or equal*** to that of the denominator, then the function is called ***improper fraction***

If the denominator can be factorised then the function can be expressed as the sum of partial fraction.

Some functions and their partial fractions are given below:

Function	*Partial fraction*
$\frac{x}{(x-1)(x+1)}$	$\frac{A}{x-1}+\frac{B}{x+1}$
$\frac{1}{(x-1)^2}$	$\frac{A}{x-1}+\frac{B}{(x-1)^2}$
$\frac{x-1}{x(x^2+1)}$	$\frac{A}{x}+\frac{Bx+C}{(x^2+1)}$
$\frac{2}{(x^2+1)^2}$	$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$

Example 1

Find $\int \frac{5}{x(1-x)}\,dx$

Solution

Example 2

Use the method of partial fraction to find $\int \frac{2x-1}{x^2-2x-3}\,dx$

Solution

Example 3

Find $\int \frac{5x^2+4x+12}{(x+2)(x^2+4)}dx$

Solution

Exercise 6

Find the following integrals by using partial fractions

a) $\int \frac{x+3}{(x+5)^2}\,dx$ *ans*: $\ln|x+5| + \frac{2}{x+5} + c$

b) $\int \frac{4x+3}{(x-2)(3+2x^2)}\,dx$ *ans*: $\ln|x\text{-}2| - \frac{1}{2}\ln|3+2x^2| + c$

LECTURE 7 OF 9
TOPIC : 1.0 INTEGRATION
SUBTOPIC : 1.4 DEFINITE INTEGRALS
LEARNING OUTCOMES : At the end of the lesson, students should be able to:
a) use the properties of definite integral
b) evaluate definite integrals.

The Properties of Definite Integral

i. $\int_a^b c\,dx = c(b-a)$, where c is any constant

ii. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

iii. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant

iv. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

v. $\int_a^a f(x)dx = 0$

vi. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$, where $a < b < c$ and

if $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

Example 1:

Given $\int_1^3 f(x)dx = 5$ and $\int_3^8 f(x) = 10$. Find

(a) $\int_3^3 f(x)dx$

(b) $\int_3^1 f(x)dx$

(c) $\int_1^3 4f(x)dx$

(d) $\int_1^8 f(x)dx$

(e) $\int_3^8 (f(x)+5)dx$

Solution:

Example 2:

Given $\int_2^5 f(x)dx = 5, \int_5^8 f(x)dx = 10$ and $\int_2^5 g(x) = 6$. Evaluate

(a) $\int_2^5 3f(x)dx$

(b) $\int_5^2 g(x)dx$

(c) $\int_2^5 [2f(x)+3g(x)]dx$

(d) $\int_2^8 f(x)dx$

Solution:

Evaluate Definite Integrals

Example 1:

Find $\int_0^2 e^{4x-2}dx$

Solution:

Example 2:

Evaluate $\int_0^1 \frac{2x^3}{\sqrt{1+x^4}}dx$

Solution:

Example 3:

Find $\int_0^{\frac{\pi}{4}} \tan 3x dx$

Solution:

Example 4:

Find $\int_1^{e} \ln 2x dx$

Solution:

Example 5:

Evaluate

$$\int_0^3 \frac{dx}{x^2+3x-4}$$

Solution:

Example 6:

Evaluate $\int_0^{\frac{\pi}{2}} (\cos 4x + \sin x)dx$

Solution:

Exercise 7

1. Given $\int_2^7 f(x)dx = 5$, find the value of

   (a) $\int_2^7 3f(x)dx$ (c) $\int_2^7 (x-2f(x))dx$

   (b) $\int_2^7 (f(x)+2)dx$ (d) $\int_7^2 (f(x)-x^2)dx$

2. Evaluate

   (a) $\int_1^3 (x+3)dx + \int_0^1 e^x dx$ (b) $\int_3^3 \sin x + \cos x dx$

3. Evaluate the following

   (a) $\int_1^2 \frac{e^{-x}+e^x}{e^x}dx$ (c) $\int_2^1 (\ln x)x dx$

   (b) $\int_2^3 \frac{x-3}{1-x}dx$ (d) $\int_0^4 (6x+1)(3x^2+x)^3 dx$

Answers:

1.	a) 15	b) 15	c) $\frac{25}{2}$	d) $\frac{320}{3}$
2.	a) 9 + e	b) 0		
3.	a) 1.059	b) 0.3863	c) -0.6363	d) 1,827,904

LECTURE 8 OF 9
TOPIC : 1.0 INTEGRATION
SUBTOPIC : 1.4 DEFINITE INTEGRALS
LEARNING OUTCOMES : At the end of the lesson, students should be able to determine the area of a region.

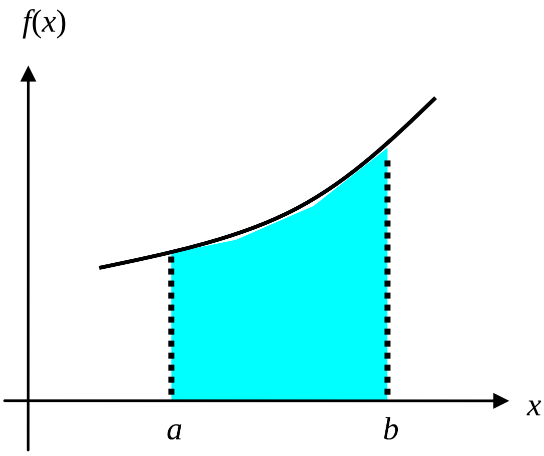
(a) Area of a Region Bounded By The Curve And The *x* – Axis

Area under a curve can be computed using definite integration. Hence the formal definition:

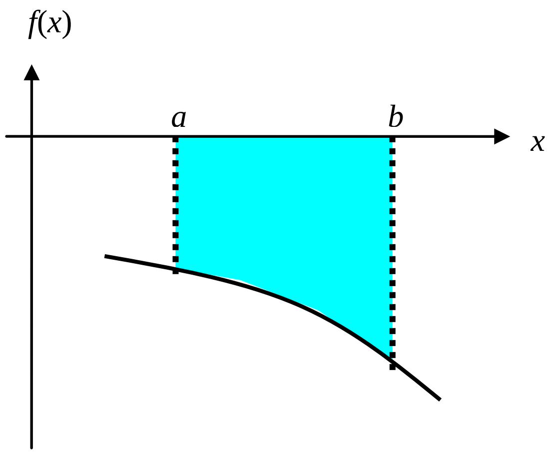
If f is continuous throughout $[a, b]$, then the area of the region between the curve $y = f(x)$ and the x – axis from $x = a$ to $x = b$ is given by

$$\textbf{area} = \int_a^b f(x)\ dx$$

Notes : Area is always positive



area = $\int_a^b f(x)dx$



area = $\left|\int_a^b f(x)dx\right|$

Example 1:

Find the area of the region bounded by the line $y = 2x$, x-axis and the line $x = 4$.

Solution:

Example 2:

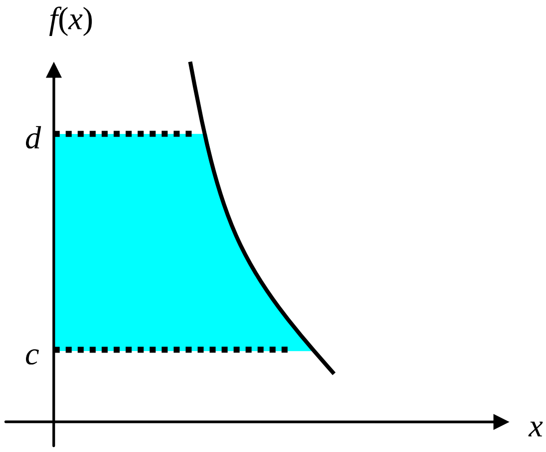
Find the area of region bounded by $f(x) = x^2 - 2x$, x-axis, the lines $x = 0$ and $x = 3$.

Solution:

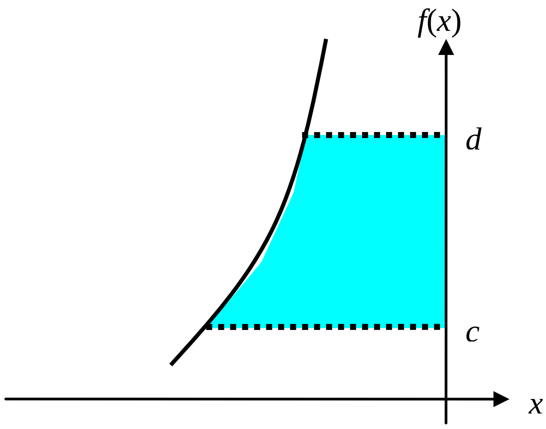
(b) Area of a Region Bounded By The Curve And The y – Axis

If g is continuous throughout $[c\,,\,d\,]$, then the area of the region between the curves $x = g(y)$ and the y – axis from $y = c$ to $y = d$ is given by

$$\textbf{area} = \int_c^d g(y)\ \ dy$$



$$\text{area} = \int_c^d g(y)dy$$



$$\text{area} = \left|\int_c^d g(y)dy\right|$$

Example 3:

Find the area enclosed by $y^2 = 4x$, y-axis, the lines $y = -2$ and $y = 2$.

Solution:

Example 4:

Find the area of region bounded by $y = \frac{1}{x}$, $2 \le y \le 4$, $-4 \le y \le -2$ and y axis.

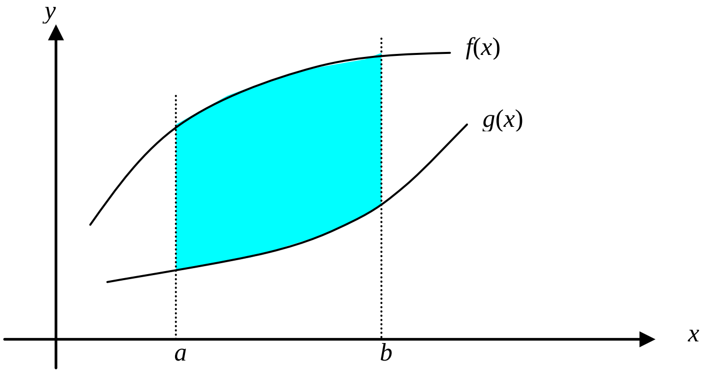
Solution:

(c) **Area of a Region Bounded By Two Curves**

Definition

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is the integral of $[f - g]$ from a to b.

$$\textbf{area} = \int_a^b [f(x) - g(x)]dx$$



Example 5:

Find the area of the region bounded by the curve $y = x^2 - 4x$ and the line $y = \frac{x}{2}$.

Solution:

Example 6:

Calculate the area of the region bounded by the curve $y^2 = x$ and $y = x - 2$.

Solution:

Example 7:

Calculate the area of the region bounded by the curve $y = x^2 + 3$, $3y = -2x + 14$ and x-axis.

Solution:

Exercise 8

1. Find the area enclosed by the curve $y = \ln x$ the line $y = 1$ and x and y axes.
2. Show that the area enclosed by the line $y = 4 - 2x$ and curve $y = 4 - x^2$ is $\frac{4}{3} unit^2$
3. Find the area of the region bounded by the graphs $y^2 = 8 - x$ and $y^2 = x$.
4. Find the area of the region bounded by the curves $y = 2\sqrt{x} + 1$ and $y = \sqrt{x}$ between $x = 0$ and $x = 4$.

Answer:

1. $(e - 1) unit^2$
2. $\frac{4}{3} unit^2$
3. $\frac{64}{3} unit^2$
4. $\frac{28}{3} unit^2$

LECTURE 9 OF 9

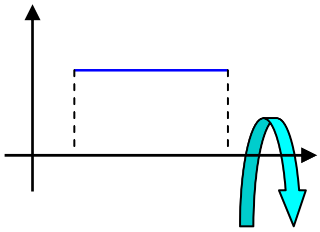
TOPIC : 1.0 INTEGRATION

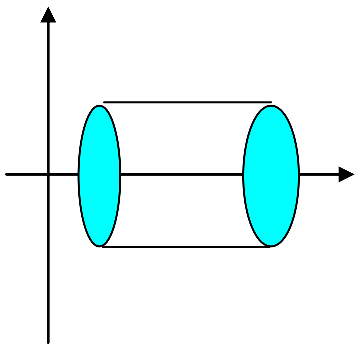
SUBTOPIC : 1.4 DEFINITE INTEGRALS

LEARNING OUTCOMES : At the end of the lesson, students should be able to find the volume of a solid of revolution.

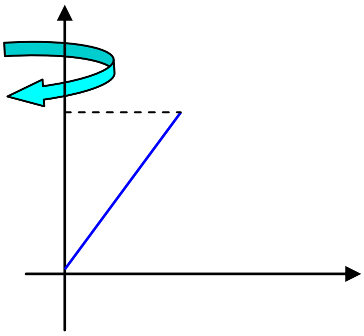
Volume of Solids Generated By Revolving The Region Bounded By The Curve And The *x* or *y* Axis

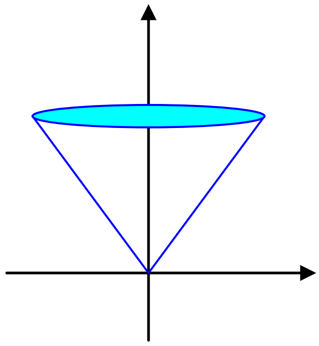
In this section we use integration to calculate the volumes of solids. Consider the solids of revolution in the following figures, which are formed by revolving certain regions about an axis.



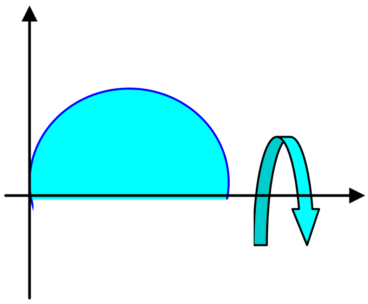


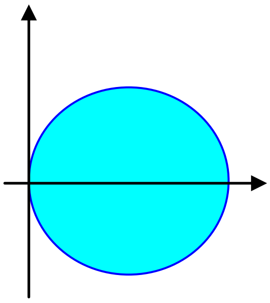
cylinder





cone





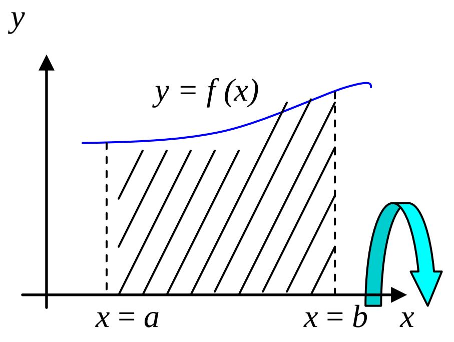
sphere

(a) Revolve about the x–axis

The volume of the solid generated by revolving the region about the x-axis between the graph of a continuous function $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is

$$\text{Volume, } V = \int_a^b \pi \, y^2 \, dx$$

$$\boxed{V = \int_a^b \pi \left[f(x) \right]^2 dx}$$

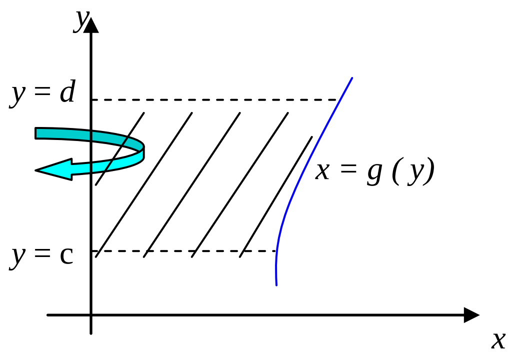


(b) Revolve about the y–axis

The volume of the solid generated by revolving the region about the y axis between the graph of a continuous function $x = g(y)$ and the y-axis from $y = c$ to $y = d$ is

$$\text{Volume, } V = \int_c^d \pi \, x^2 \, dy$$

$$\boxed{V = \int_c^d \pi \left[g(y) \right]^2 dy}$$



Example 1:

Find the volume of the solid formed when the area bounded by the curves $\boldsymbol{y} = \frac{1}{\boldsymbol{x}}$, *y*-axis, $y = 2$ and $y = 5$ is rotated about *y*-axis.

Solution:

Example 2:

Let R be the region bounded by the curve $y = -x^2 + 1$. Calculate the volume of the solid formed when it is rotated about $x - axis$

Solution:

Example 3:

Let R be the region bounded by curve $y = e^x$, and $2 \le x \le 5$. Calculate the volume of the solid formed when R is rotated about the x-axis.

Solution:

Example 4:

The area R bounded by the curves $y=\sqrt{x-2}$, and between $0 \le y \le 2$ is revolved completely about $y-axis$. Calculate the volume of the solid formed.

Solution:

Volume of Solids Generated By Revolving The Region Bounded By Two Curves

(a) Revolve About *x* Axis

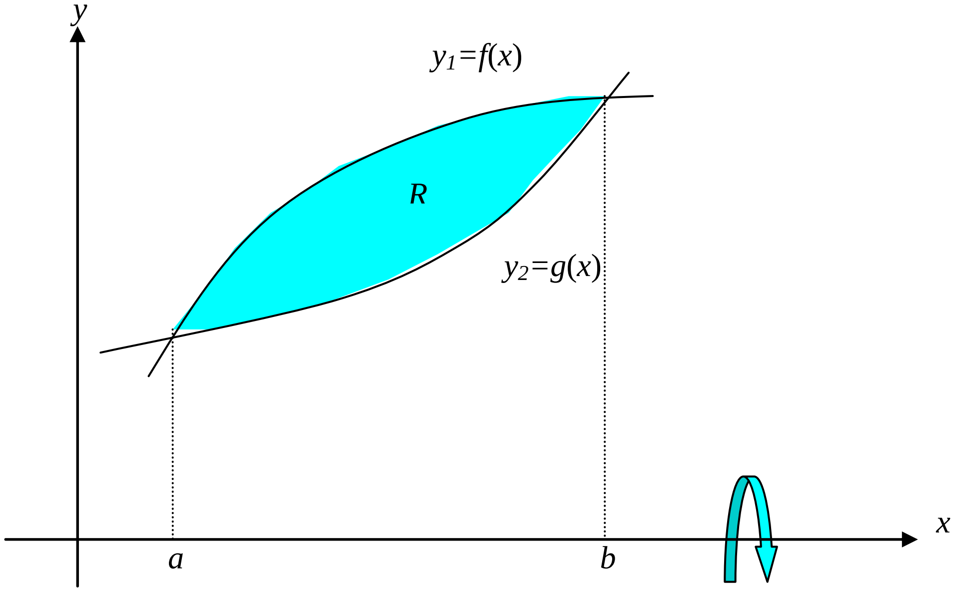
Suppose that $y_1 = f(x)$ and $y_2 = g(x)$ are nonnegative continuous function such that , $f(x) \geq g(x)$ for $a \leq x \leq b$ and let R be the region enclosed between the graphs of these functions and the lines $x = a$ and $x = b$. When this region is revolved about the x-axis, it generates a solid .

The volume of the solid generated by revolving R about x-axis is

$$V = \int_a^b \pi \left(y_1^2 - y_2^2 \right) dx$$

or

$$\boxed{V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx}$$



(b) Revolve About y-Axis

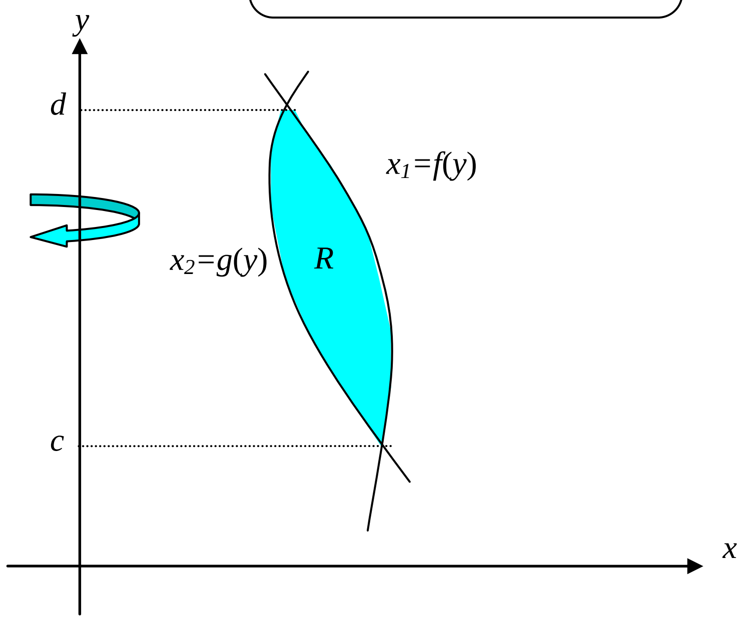
Suppose that $x_1 = f(y)$ and $x_2 = g(y)$ are nonnegative continuous function such that , $f(y) \geq g(y)$ for $c \leq y \leq d$ and let R be the region enclosed between the graphs of these functions and the lines $y = c$ and $y = d$. When this region is revolved about the y- axis, it generates a solid.

The volume of the solid generated by revolving R about the y-axis is

$$V = \int_c^d \pi\left(x_1^{\ 2} - x_2^{\ 2}\right)dy$$

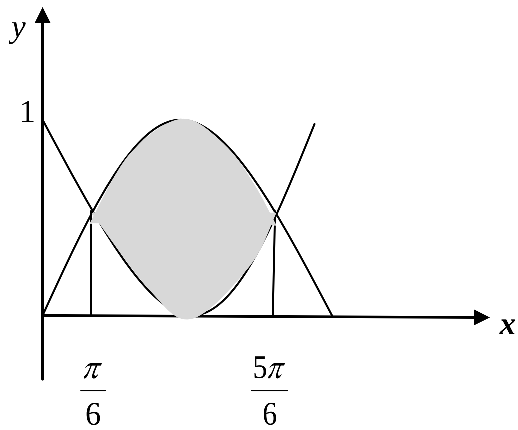
or

$$\boxed{V = \int_c^d \pi\left([f(y)]^2 - [g(y)]^2\right)dy}$$



Example 5:

Find the volume of the solid generated when the area bounded by $y = \sin x$ and $y = 1 - \sin x$ for $0 \le x \le \pi$ is rotated 360° about the x-axis.



Solution:

Exercise 9

1. The region bounded by $y = \tan x$, and between $0 \le x \le \frac{\pi}{3}$ .Calculate the volume formed when the region is rotated about x - axis.
2. Let R be the region bounded by $f(x) = |x-4|$, $0 \le x \le 4$, and y - axis. Find the volume of the solid generated by revolving the region R through 360° about y -axis.
3. Region R is bounded by the curve $x = \sqrt{y} + 3, x = 3$ *and* $y = 4$. Find the volume of the solid formed when this region is rotated through 2π about y - axis.
4. Calculate the volume of the solid when the region formed by $y = x^2$, $y = \frac{1}{x}$ and the line $x = \frac{1}{2}$ is rotated about x-axis.

Answer:

1. $\pi\left[\sqrt{3} - \frac{\pi}{3}\right] unit^3$
2. $\frac{64}{3}\pi\, unit^3$
3. $40\pi\, unit^3$
4. $\frac{129}{160}\pi\, unit^3$