1. If $\log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$, prove that $x^2 + y^2 = 7xy$. **[5 marks]**

2. Find the non-zero value of x that satisfies the equation $3^{2x} - 3^{x+1} + 2 = 0$. Give your answer correct to three decimal places.

[7 marks]

3. Solve the inequality $\left|\frac{x+1}{x-4}\right| \geq 2$. **[7 marks]**

4. If $z = 2 - i$, write the complex number $\frac{z + 5\bar{z}}{3 - z}$ in the form $a + bi$, where a and b are real numbers. Hence, find the polar form of $\frac{z + 5\bar{z}}{3 - z}$. **[7 marks]**

5. a) Find the least number of terms of the geometric sequence

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \ldots$$

which must be taken so that the sum of the terms exceeds 99.99% of the sum to infinity of the series.

[6 marks]

b) Given that the 4^{th} terms in the expansion of $\left(px + \frac{1}{x}\right)^n$ is $\frac{5}{2}$. Find n and p given n is positive integer.

[6 marks]

6. Given the matrix $A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

a) Find $|A|$ and adjoint A. Hence determine A^{-1}. **[8 marks]**

b) Using the result from part (a), solve the following system of linear equations.

$$2x - y + 2z = 3$$
$$3y + z = 2$$
$$x + 2y + 3z = -2$$

[4 marks]

Final Answers

1. prove

2. 0.631

3. $\left\{x : \frac{7}{3} \le x < 4 \;\; or \;\; 4 < x \le 9\right\}$

4. $8 - 4i$; $4\sqrt{5}\left(\cos(-0.4636) + i\sin(-0.4636)\right)$

5. a) 14

   b) $n = 6$ and $p = \frac{1}{2}$

6. a) $|A| = 7$; $adj(A) = \begin{bmatrix} 7 & 7 & -7 \\ 1 & 4 & -2 \\ -3 & -5 & 6 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{1}{7} & \frac{4}{7} & \frac{-2}{7} \\ \frac{-3}{7} & \frac{-5}{7} & \frac{6}{7} \end{bmatrix}$

   b) $x = 7$, $y = \frac{15}{7}$, $z = -\frac{31}{7}$

Answer Scheme

1. $\log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$

$$= \log 3 + \log x^{\frac{1}{2}} + \log y^{\frac{1}{2}}$$

$$= \log\left(3x^{\frac{1}{2}}y^{\frac{1}{2}}\right)$$

$$x + y = 3x^{\frac{1}{2}}y^{\frac{1}{2}}$$

$$(x+y)^2 = 9xy$$

$$x^2 + 2xy + y^2 = 9xy$$

$x^2 + y^2 = 7xy$. Proved

2. $3^{2x} - 3^{x+1} + 2 = 0$

$$(3^x)^2 - 3(3^x) + 2 = 0$$

let $u = 3^x$

$$u^2 - 3u + 2 = 0$$

$$(u-2)(u-1) = 0$$

$u = 2$	or	$u = 1$
$3^x = 2$	or	$3^x = 1$
$x\ln 3 = \ln 2$		$x \ln 3 = \ln 1$
$x = \frac{\ln 2}{\ln 3}$		$x = 0$

$\therefore \quad x = 0.631$.

3. $\left|\frac{x+1}{x-4}\right| \geq 2$.

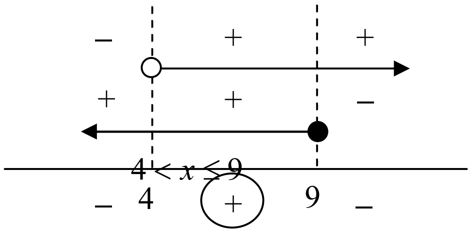
$\frac{x+1}{x-4} \ge 2$ or $\frac{x+1}{x-4} \le -2$

$\frac{x+1}{x-4} - 2 \ge 0$ $\frac{x+1}{x-4} + 2 \le 0$

$\frac{-x+9}{x-4} \ge 0$ $\frac{3x-7}{x-4} \le 0$

Let $-x+9 \ge 0$, $x-4>0$ Let $3x-7 \ge 0$, $x-4>0$

$x \le 9$, $x>4$ $x \ge \frac{7}{3}$, $x>4$



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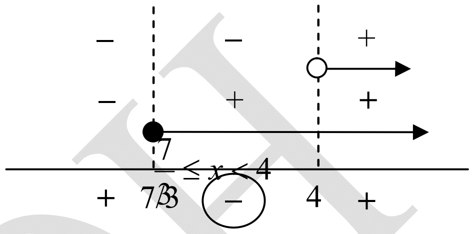

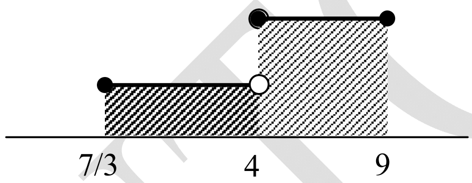

$\therefore \quad \frac{7}{3} \le x < 4 \quad \cup \quad 4 < x \le 9$

4. $\frac{z+5\bar{z}}{3-z} = \frac{(2-i)+5(2+i)}{3-(2-i)}$

$= \frac{12+4i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{16-8i}{2}$

$= 8-4i$

$\left|\frac{z+5\bar{z}}{3-z}\right| = \sqrt{(8)^2 + (-4)^2}$

$= \sqrt{80}$

$= 4\sqrt{5}$

$\alpha = \tan^{-1}\left|\frac{-4}{8}\right|$

$= 0.4636 \text{ rad}$

$\theta = -0.4636 \text{ rad}$

$$\therefore \frac{z+5\bar{z}}{3-z} = 4\sqrt{5}\left(\cos(-0.4636) + i\sin(-0.4636)\right)$$

5. a) $1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\ldots$ is GS with $a=1$ and $r=\frac{1}{2}$

$$S_n > \frac{99.99}{100}S_\infty$$

$$\frac{a(1-r^n)}{1-r} > \frac{99.99}{100}\left(\frac{a}{1-r}\right)$$

$$(1-r^n) > 0.9999$$

$$r^n < 0.0001$$

$$\left(\frac{1}{2}\right)^n < 0.0001$$

$$n\ln\left(\frac{1}{2}\right) < \ln(0.0001)$$

$$n > \frac{\ln(0.0001)}{\ln\left(\frac{1}{2}\right)}$$

$$n > 13.29$$

$\therefore\ n = 14$.

b) In the expansion of $\left(px+\frac{1}{x}\right)^n$,

$$T_4 = \frac{5}{2}$$

$$\binom{n}{3}(px)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2}$$

$$\binom{n}{3}p^{n-3}x^{n-6} = \frac{5}{2}x^0$$

$$n-6=0 \quad \Rightarrow \quad n=6$$

and $\binom{6}{3}p^3 = \frac{5}{2}$

$$p^3 = \frac{5}{2}\times\frac{1}{20} = \frac{1}{8} \quad \Rightarrow \quad p = \frac{1}{2}$$

$\therefore\ n=6$ and $p=\frac{1}{2}$.

6. (a) $|A| = \begin{vmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} - 0 + \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 2(9-2)+(-1-6) = 7$.

$adj(A) = \left[C_{ij}\right]^T$ and $C_{ij} = (-1)^{i+j}M_{ij}$

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 1 & -3 \\ 7 & 4 & -5 \\ -7 & -2 & 6 \end{bmatrix}$$

$$\therefore \qquad adj(A) = \begin{bmatrix} 7 & 7 & -7 \\ 1 & 4 & -2 \\ -3 & -5 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{7}\begin{bmatrix} 7 & 7 & -7 \\ 1 & 4 & -2 \\ -3 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{1}{7} & \frac{4}{7} & -\frac{2}{7} \\ -\frac{3}{7} & -\frac{5}{7} & \frac{6}{7} \end{bmatrix}.$$

(b)

$$2x - y + 2z = 3$$
$$3y + z = 2$$
$$x + 2y + 3z = -2$$

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 7 & 7 & -7 \\ 1 & 4 & -2 \\ -3 & -5 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 49 \\ 15 \\ -31 \end{bmatrix} = \begin{bmatrix} 7 \\ \frac{15}{7} \\ -\frac{31}{7} \end{bmatrix}$$

$\therefore \qquad x = 7$, $y = \frac{15}{7}$ and $z = -\frac{31}{7}$.