LECTURE 1 OF 6

TOPIC : 8.0 LIMITS AND CONTINUITY

SUBTOPIC : 8.1 Limits

OBJECTIVES : At the end of the lesson students are able to:

a) state limit of a function *f(x)* as *x* approaches a given value *a*, $\lim_{x\to a} f(x) = L$.

b) apply the basic properties of limit.

CONTENT :

8.1.a Definitions of Limit

A function *f(x)* is said to approach a constant *L* as a limit when *x* approaches *a* as below,

$$\lim_{x\to a} f(x) = L$$

where *f(x)* is the function which assumes a corresponding set of values and *x* is an independent variable.

A key point with the limit concept is that we are not interested in the value of *f(x)* when $x = a$. We are interested in the behavior of *f(x)* as *x* comes closer and closer to a value of *a*. And the notation

$$\lim_{x\to a} f(x) = L$$

means that as *x* gets close to *a*, but $x \neq a$, *f(x)* gets close to *L*.

Example 1

Determine $\lim_{x\to 0}\left(x^2+1\right)$

8.1b Properties of Limits

i. If $f(x) = c$, where c is a constant, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} c = c$$

Example 2

$$\lim_{x\to 3} 5 =$$

ii. If $f(x) = x$, then

$$\lim_{x\to a} x = a$$

Example 3

$$\lim_{x\to 4} x =$$

iii. If $f(x) = x^n$, where n is a positive integer (n > 0), then

$$\lim_{x\to a} \left(x^n\right) = a^n$$

Example 4

$$\lim_{x\to 4}\left(x^3\right)=$$

iv. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

Example 5

$$\lim_{x\to -1}(x^3-10)=$$

v. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a}\left[f(x).g(x)\right]=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

Example 6

$$\lim_{x\to 4}(x^2)(x+1) =$$

vi. If c is a constant, then

$$\lim_{x\to a}\left[cf(x)\right]=c.\lim_{x\to a} f(x)$$

Example 7

$$\lim_{x\to 2}\left(5x^3\right)=$$

vii. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad , \lim_{x\to a} g(x) \neq 0$$

Example 8

$$\lim_{x\to -5}\frac{x}{x^2+10}=$$

LECTURE 2 OF 6

TOPIC : 8.0 LIMITS AND CONTINUITY

SUBTOPIC : 8.1 Limits

**OBJECTIVES : ** At the end of the lesson students are able to:

c) find $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ when $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$,

by following methods:
i. factorization
ii. multiplication of conjugates

CONTENT :

Limit of the Rational Function:

The limit of a rational function can be found by substitution when the denominator is not equal to zero.

If *f(x)* and *g(x)* are polynomials, and c is any number, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \qquad \text{provided} \qquad g(c) \neq 0$$

Example 1

Find $\lim_{x\to 2} \dfrac{x^2+4}{x+2}$

If the solution for limit problem of a rational function is $\dfrac{0}{0}$, we cannot calculate the limit of the given rational function by substitution. We need to simplify the fraction by

(i) factorization
(ii) multiplication of conjugates

(i) Factorization Method

Example 2

Find the $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$

Example 3

Find the $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$

(ii) Multiplication of Conjugates Method.

If the solution for limit problem of a rational function related to surd using substitution method equal to $\frac{0}{0}$ then use the multiplication of conjugates method.

Example 4

Find the limits below:

(i) $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

(ii) $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

LECTURER 3 OF 6

TOPIC : 8.0 LIMITS AND CONTINUITY

SUBTOPIC : 8.1 Limits

OBJECTIVES: At the end of the lesson students are able to:

d) find one-sided limits
   i. $\lim_{x\to a^+} f(x) = L$
   ii. $\lim_{x\to a^-} f(x) = M$

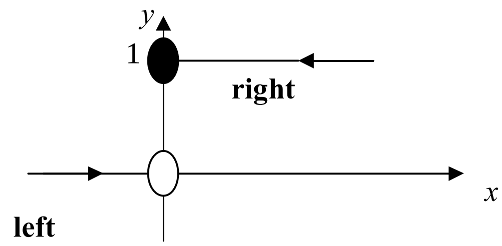
e) determine the existence of the limit of a function
   $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

f) find infinite limits
   i. $\lim_{x\to a} f(x) = +\infty$
   ii. $\lim_{x\to a} f(x) = -\infty$

Sketch of graphs may be necessary to explain (i) and (ii) above.

CONTENT:

One sided limits:



We noticed that $f(x)$ approaches 0 as x approaches from the left and $f(x)$ approaches 1 as x approaches 0 from the right.

A one - sided limit can either be a right-hand limit or a left ï hand limit.

Right- hand limit:

$$\lim_{x\to a^+} f(x) = L$$

i.e. the limit of f as x approaches a from the right is L

Left- hand limit:

$$\lim_{x\to a^-} f(x) = M$$

i.e. the limit of f as x approaches a from the left is M

One-sided limits are useful in taking limits of functions involving radicals.

Example 1

Find $\lim_{x \to 1} \sqrt{x-1}$

Example 2

Find the $\lim_{x \to 2} \sqrt{4-x^2}$

Test for Existence of a Limit

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

then $\lim_{x \to a} f(x) = L$

Example 3

Given the function,

$$f(x) = \begin{cases} 2x, & x \leq 4 \\ 2x+3, & x > 4. \end{cases}$$

Determine $\lim_{x \to 4} f(x)$ if it exist.

Example 4

Given $f(x) = \begin{cases} 2 & ; \quad x \neq 1 \\ 5 & ; \quad x = 1 \end{cases}$, find $\lim_{x \to 1} f(x)$.

Example 5

Given $f(x) = |x-2|$. Find $\lim_{x \to 2} f(x)$

Find Infinite Limits

i) $\lim_{x \to a} f(x) = +\infty$

ii) $\lim_{x \to a} f(x) = -\infty$

Example 6

Given $f(x) = \frac{1}{x-1}$. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

Example 7

Given $f(x) = \frac{1}{(x-2)^2}$,find $\lim_{x\to 2^+} \frac{1}{(x-2)^2}$ and $\lim_{x\to 2^-} \frac{1}{(x-2)^2}$

LECTURE 4 OF 6

TOPIC : 8.0 LIMITS AND CONTINUITY

SUBTOPIC : 8.1 Limits

OBJECTIVES : At the end of the lesson student s are able to:

g) find limits at infinity
  i. $\lim_{x\to +\infty} f(x) = L$
  ii. $\lim_{x\to -\infty} f(x) = M$

h) find $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ when $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are undefined.

i) use
  i. $\lim_{x\to +\infty}\left[\frac{1}{x^n}\right] = 0$
  ii. $\lim_{x\to -\infty}\left[\frac{1}{x^n}\right] = 0$
  for $n > 0$

Limits at infinity

Example 1

Given $f(x) = \dfrac{1}{x^2}$, find $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Sketch the graph.

Notes : $\lim\limits_{x\to+\infty}\left[\dfrac{1}{x^n}\right] = 0$ or $\lim\limits_{x\to-\infty}\left[\dfrac{1}{x^n}\right] = 0$

Example 2

Find the limits at infinity for the following :

(a) $\lim\limits_{x\to+\infty}\left(x^2+3\right)$ (b) $\lim\limits_{x\to-\infty}\left(1-\dfrac{5}{x}\right)$ (c) $\lim\limits_{x\to+\infty}\dfrac{2}{(x-3)^3}$

Limits at Infinity (for Rational Function)

If $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \dfrac{\infty}{\infty}$, divide each term by x to the highest power of the denominator.

Example 3

Find the limit of the following **:**

i. $\lim\limits_{x\to\infty}\dfrac{2x^3+x^2-3}{x^3+x+2}$

ii. $\lim\limits_{x\to-\infty}\left(\dfrac{x}{3x^2-1}\right)$

iii. $\lim\limits_{x\to-\infty}\dfrac{x+1}{\sqrt{x^2+1}}$

iv. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+4}}{x+3}$

LECTURE 5 OF 6

TOPIC : 8.0 LIMITS AND CONTINUITY

SUBTOPIC : 8.2 Asymptotes

OBJECTIVE : At the end of the lesson, students are able to :

a) determine vertical and horizontal asymptotes

CONTENT :

Definition of Vertical Asymptote

A line x = a is a vertical asymptote of the graph of a function y = f(x) if either

$$\lim_{x\to a^-} f(x) = +\infty \quad (\text{or } -\infty)$$

OR

$$\lim_{x\to a^+} f(x) = +\infty \quad (\text{or } -\infty)$$

Example 1

Find the vertical asymptote of $f(x)=\dfrac{1}{x^2}$.

Example 2

Determine the vertical asymptote of

i) $f(x)=\dfrac{x+3}{x+2}$

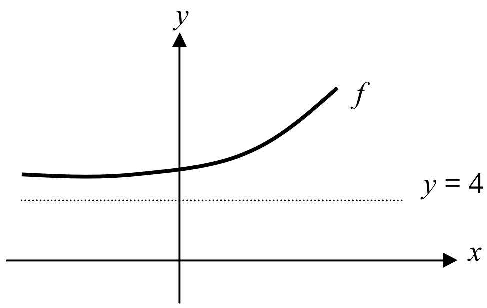
ii) $f(x)=\dfrac{1}{x^2-1}$

iii) $f(x)=\dfrac{2x}{x^2-x-2}$

Definition of horizontal asymptote

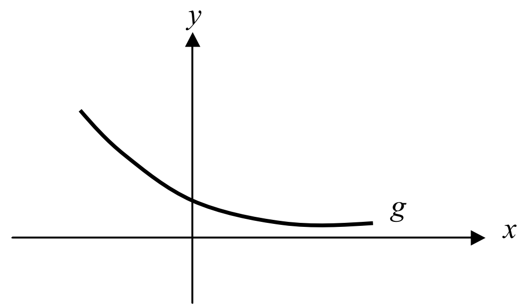
A line y = b is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b$$



$$\lim_{x \to -\infty} f(x) = 4$$

Fig. 8.2.1



$$\lim_{x \to \infty} g(x) = 0$$

Fig. 8.2.2

Examine the two functions sketched in Fig. 8.2.1 and Fig. 8.2.2. In Fig. 8.2.1, as x approaches $-\infty$, $f(x)$ approaches but never quite reaches a value of 4. Using limit notation, we state

$$\lim_{x \to -\infty} f(x) = 4$$

Hence, $f(x)$ has a horizontal asymptote of $y = 4$.

Similarly, in Fig. 8.2.2, g approaches but never quite reaches the x axis as x approaches ∞. We can state this behavior by the notation

$$\lim_{x \to \infty} g(x) = 0$$

Hence, $g(x)$ has a horizontal asymptote of $y = 0$.

Example 3

Determine the horizontal asymptote

a) $f(x) = \dfrac{x}{x-2}$ b) $f(x) = \dfrac{2x^2 - 1}{x^2 + 1}$

LECTURE 6 OF 6

TOPIC : 8.0 LIMITS AND CONTINUITY

SUBTOPIC : 8.3 Continuity

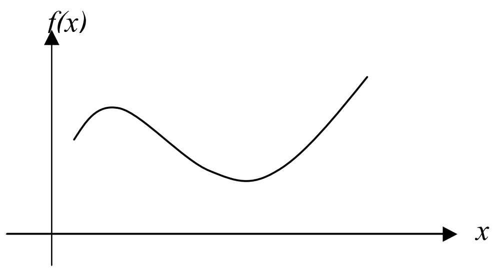
OBJECTIVES : At the end of the lesson, students are able to :

a) discuss the continuity of a function at a point
b) determine the continuity of a function at a point

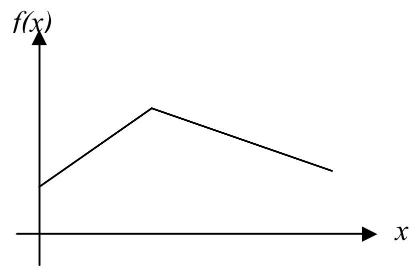
CONTENT

The Definition of Continuity

A function $y = f(x)$ that can be graphed throughout its domain with one continuous motion of the pen (that is, without lifting the pen) is an example of a continuous function.

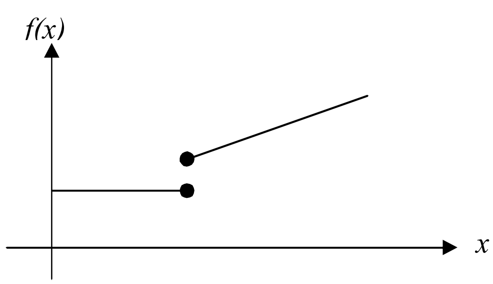


(i) continuous

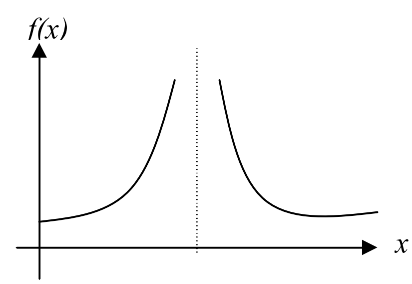


(ii) continuous

Figure (i) and (ii) continuous since they can be drawn without lifting your pencil.



(iii) discontinuous



(iv) discontinuous

Figure (iii) and (iv) are not continuous because of the ñ breaksò in the function.

Definition of Continuity at a point

A function $y = f(x)$ is continuous at point $x=c$ if and only if it meets **all** three of the following conditions.

(i) $f(c)$ exists (f is defined at c)

(ii) $\lim_{x\to c} f(x)$ exists $\Leftrightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$ (f has a limit as $x \to c$)

(iii) $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

Discontinuity at a Point

If a function f is not continuous at a point c, we say that f is discontinuous at c and or c is a point of discontinuity of f.

Example 1

Let $f(x) = \begin{cases} x^2 + 1 & , \ x < 2 \\ x + 3 & , \ x \geq 2 \end{cases}$

Is $f(x)$ continuous at $x = 2$

Example 2

Let $h(x) = \begin{cases} 20x^2 - k & , \ x < 4 \\ p & , \ x = 4 \\ kx^3 - 5 & , \ x > 4 \end{cases}$

Find the value of k such that $\lim_{x\to 4} h(x)$ exists. Hence, find the value of p such that h is continuous at $x = 4$

Example 3

Given that $f(x) = \begin{cases} ax+5, & x \leq 1 \\ x^2 - 1, & x > 1 \end{cases}$

If $f(x)$ is continuous for all values of x, find the values of the constant a.