LECTURE 1 OF 5

TOPIC : 7.0 TRIGONOMETRIC FUNCTIONS

SUBTOPIC : 7.1 Trigonometric Ratios and Identities

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:

a) state trigonometric ratios of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\text{cosec}\,\theta$, $\sec\theta$ and $\cot\theta$.

b) use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\sin(90^o-\theta) = \cos\theta$, $\cos(90^o-\theta) = \sin\theta$, $\tan(90^o-\theta) = \cot\theta$.

c) use some special angles.

d) evaluate trigonometric functions for any angle

e) use the Pythagorean identities :

$$\sin^2\theta + \cos^2\theta = 1$$

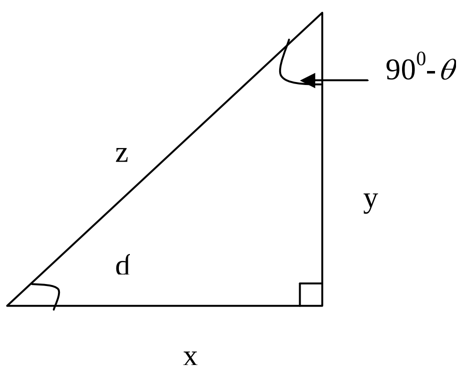
$$1 + \cot^2\theta = \text{cosec}^2\,\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

CONTENT

7.1 a) Trigonometric Ratios of sinθ , cosθ, tanθ, cosecθ , sec θ and cot θ

For any acute angle θ, there are six trigonometric ratios, each of which is defined by referring to a right angled triangle containing θ.



From the diagram

$\sin\theta = \frac{y}{z}$ cosec θ = $\frac{1}{\sin\theta} = \frac{z}{y}$

cos θ = $\frac{x}{z}$ sec θ = $\frac{1}{\cos\theta} = \frac{z}{x}$

tan θ = $\frac{y}{x}$ cot θ = $\frac{1}{\tan\theta} = \frac{x}{y}$

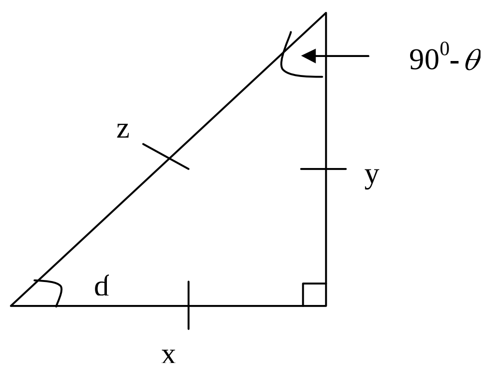
(Note : $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$)

Example 1

Given $\tan\theta = \frac{1}{\sqrt{7}}$ and θ is an acute angle, evaluate

$$\frac{\sin^2\theta}{4+\cos^2\theta}$$

7.1 b) Relationship of sin θ, cos θ and tan θ



From the diagram

a) $\sin(90^0 - \theta) = \frac{x}{z}$ $\cos\theta = \frac{x}{z}$

Therefore $\boxed{\sin(90^0 - \theta) = \cos\theta}$

b) $\cos(90^0 - \theta) = \frac{y}{z}$ $\sin\theta = \frac{y}{z}$

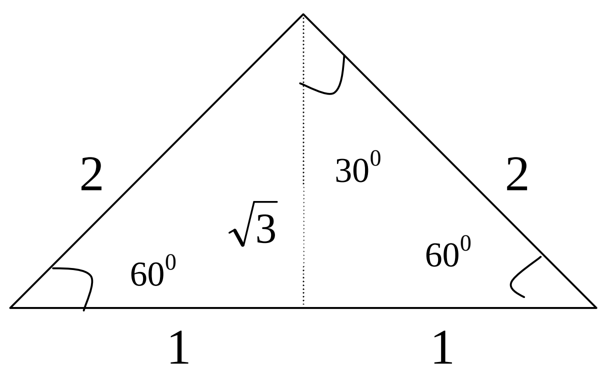
Therefore $\boxed{\cos(90^0 - \theta) = \sin\theta}$

c) $\tan(90^0 - \theta) = \dfrac{x}{y}$ $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\frac{y}{x}} = \dfrac{x}{y}$

Therefore $\boxed{\tan(90^0 - \theta) = \cot\theta}$

7.1 c) Trigonometric Ratios of Particular Angles

Equilateral triangle of sides 2 unit in length

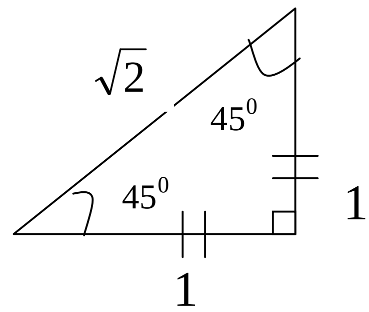


$\sin 60^0 = \dfrac{\sqrt{3}}{2}$ $\sin 30^0 = \dfrac{1}{2}$

$\cos 60^0 = \dfrac{1}{2}$ $\cos 30^0 = \dfrac{\sqrt{3}}{2}$

$\tan 60^0 = \dfrac{\sqrt{3}}{1}$ $\tan 30^0 = \dfrac{1}{\sqrt{3}}$

Isosceles triangle



Hence,

$$\sin 45^0 = \frac{1}{\sqrt{2}} \quad , \quad \cos 45^0 = \frac{1}{\sqrt{2}} \quad , \quad \tan 45^0 = 1$$

The values of trigonometric ratio for some particular angles are as follows:

θ	0^0 (0 rad)	30^0 $\left(\frac{\pi}{6} \text{ rad}\right)$	45^0 $\left(\frac{\pi}{4} \text{ rad}\right)$	60^0 $\left(\frac{\pi}{3} \text{ rad}\right)$	90^0 $\left(\frac{\pi}{2} \text{ rad}\right)$
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\frac{\sqrt{3}}{1}$	∞

Example 2

Without using table or calculator, show that

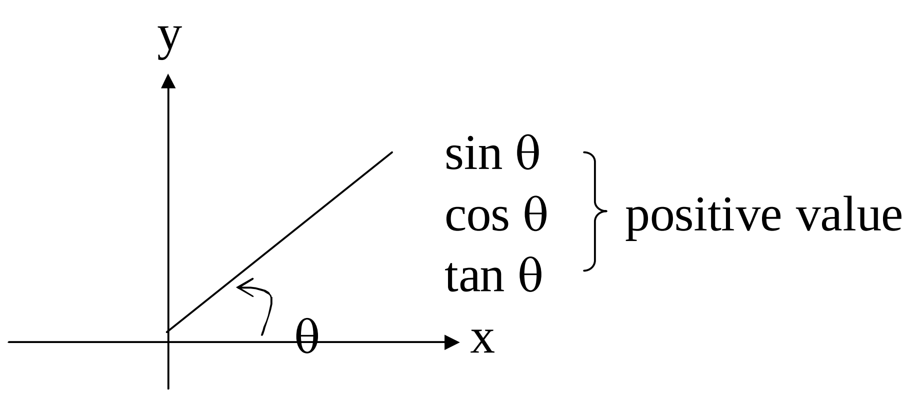
a) $\cos 60^0 = \dfrac{1 - \tan^2 30^0}{1 + \tan^2 30^0}$

b) $\sec 30^0 \tan 60^0 + \sin 45^0 \operatorname{cosec} 45^0 +$
$\cos 30^0 \cot 60^0 = \dfrac{7}{2}$

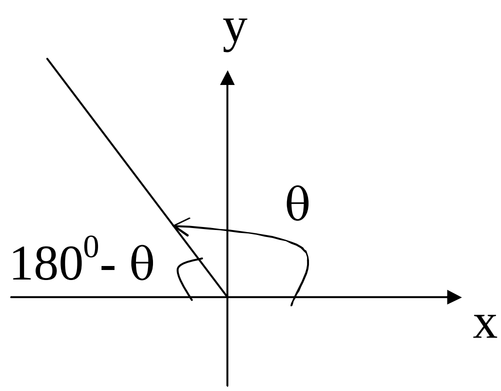
7.1 d) Trigonometric Ratio for Any Angle

i) Positive Angle

First Quadrant

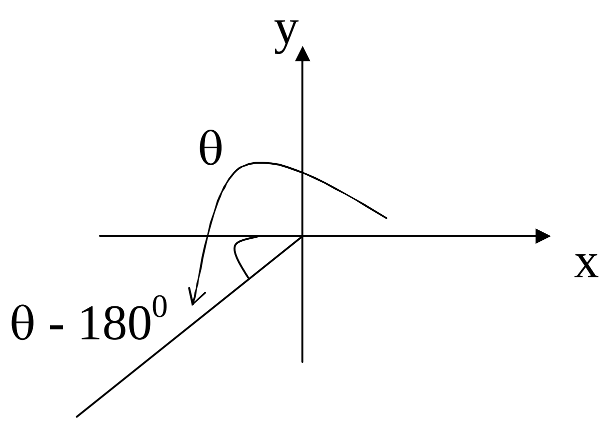


Second Quadrant



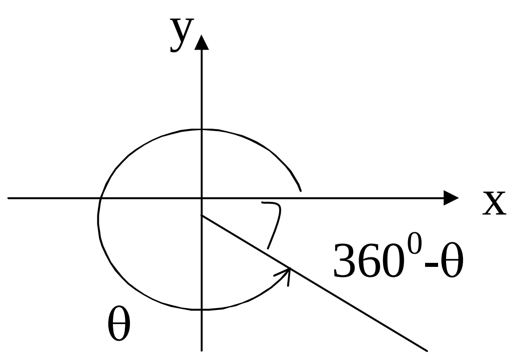
$\sin\theta = \sin(180^0 - \theta)$
$\cos\theta = -\cos(180^0 - \theta)$
$\tan\theta = -\tan(180^0 - \theta)$

Third Quadrant



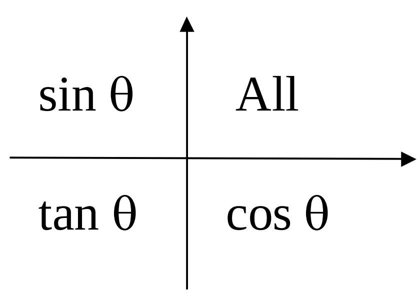
$\sin\theta = -\sin(\theta - 180^0)$
$\cos\theta = -\cos(\theta - 180^0)$
$\tan\theta = \tan(\theta - 180^0)$

Fourth Quadrant



$\sin\theta = -\sin(360^0 - \theta)$
$\cos\theta = \cos(360^0 - \theta)$
$\tan\theta = -\tan(360^0 - \theta)$

We can summarize that all the trigonometric ratios are positive in the first quadrant, sine is positive in second quadrant, tangent is positive in the third quadrant and cosine is positive in the fourth quadrant as shown below.



ii) Negative Angle

The rotating arm will describe a negative angle if it rotates in a clock-wise direction.

In general

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Example 3

State the trigonometric ratio in acute angle.

a) $\sin 140^0$ b) $\tan 312^0$

c) $\sin(-43^0)$ d) $\cos(-154^0)$

Example 4

Without using calculator, evaluate

a) $\cos 210^0$ b) $\tan(-120^0)$

7.1 e) Use the Pythagorean Identities

$\cos^2 \theta + \sin^2 \theta = 1$

$1 + \tan^2 \theta = \sec^2 \theta$

$\cot^2 \theta + 1 = \text{cosec}^2 \theta$

Example 5

Prove that

a) $\tan\theta + \cot\theta = \sec\theta \cos ec\theta$

b) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\tan\theta + \sec\theta)^2$

Exercises

1. Prove that

$$(\cos ec\theta - \sin\theta)(\sec\theta - \cos\theta) = \frac{1}{\tan\theta + \cot\theta}$$

2. Prove that

$$2\sin\theta\cos\theta + \sin^3\theta\sec\theta + \cos^3\theta\cos ec\theta = \tan\theta + \cot\theta$$

3. Prove that

$$\frac{1}{\cos ec\theta - \cot\theta} - \frac{1}{\sin\theta} = \frac{1}{\sin\theta} - \frac{1}{\cos ec\theta + \cot\theta}$$

LECTURE 2 OF 5

TOPIC : 7.0 TRIGONOMETRIC FUNCTIONS

SUBTOPIC : 7.2 COMPOUND ANGLE

LEARNING OUTCOMES :

At the end of the lesson, students should be able to

a) use the formulae $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$

b) use the double-angle formulae

c) use the half-angle formulae

d) use the factor formulae

i. $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

ii. $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

iii. $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

iv. $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

7.2 a) Use the formulae $\sin(A \pm B)$, $\cos(A \pm B)$ **and** $\tan(A \pm B)$

For any angles A and B.

a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

b) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

c) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

d) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

e) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

f) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 1

Without using calculators, find the value for the following in terms of surd.

a) $\sin 75^0$

b) $\tan 15^0$

Example 2

Given $\cos A = -\frac{12}{13}$, $\tan B = \frac{3}{4}$, A in second quadrant and B in third quadrant. Find the values of $\sin(A+B)$, $\cos(A+B)$ and $\tan(A+B)$.

Example 3

Prove that $\cot A + \tan 2A = \cot A \sec 2A$

7.2 b) Use The Double-Angle Formulae

For any angles A,

a) $\sin 2A = 2\sin A\cos A$

b) $\cos 2A = \cos^2 A - \sin^2 A$

$$= 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

c) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Example 4

Prove that $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$

7.2 c) Use The Half-Angle Formulae

For any angle A,

a) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

b) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1$

$= 1 - 2\sin^2\frac{A}{2}$

c) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

Example 5

Prove that $\operatorname{cosec} A - \cot A = \tan\frac{A}{2}$.

Deduce that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

7.2 d) Use The Factor Formulae

For any angles P and Q

a) $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

b) $\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

c) $\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

d) $\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

Example 6

Express each of the following as the product of two trigonometric functions.

a) $\sin 6\theta + \sin 4\theta$

b) $\cos 8\theta + \cos 4\theta$

c) $\cos 7\theta - \cos 5\theta$

d) $\sin 8\theta - \sin 2\theta$

For any angles P and Q

a) $\sin P \cos Q = \frac{1}{2}[\sin (P + Q) + \sin (P - Q)]$

b) $\cos P \sin Q = \frac{1}{2}[\sin (P + Q) - \sin (P - Q)]$

c) $\cos P \cos Q = \frac{1}{2}[\cos (P + Q) + \cos (P - Q)]$

d) $\sin P \sin Q = -\frac{1}{2}[\cos (P + Q) - \cos (P - Q)]$

Example 7

Express each of the following as the sum or difference of two trigonometric functions.

a) $\sin 6A \sin 2A$
b) $\sin 3A \cos A$
c) $\cos 4A \sin 2A$
d) $4 \cos 5A \cos 3A$

Example 8

Prove that $\dfrac{\sin 7A + \sin 3A}{\cos 7A + \cos 3A} = \tan 5A$

Exercises

1. Prove that

$$\tan A - 2\tan(A + \frac{\pi}{4}) + \tan(A + \frac{\pi}{2}) = \frac{(\tan A + 1)(\tan^2 A + 1)}{\tan A(\tan A - 1)}$$

2. If $A + B + C = 180^0$, prove that

$$\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$$

3. Show that for all angle A, B and C,

$$\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A \cos B \cos C - 1$$
$$= -4\sin\frac{(A+B+C)}{2}\sin\frac{(B+C-A)}{2}\sin\frac{(C+A-B)}{2}\sin\frac{(A+B-C)}{2}$$

LECTURE 3 OF 5

TOPIC :7.0 TRIGONOMETRIC FUNCTIONS

SUBTOPIC : 7.3 Solution of Trigonometric Equations

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:

a) solve trigonometric equations such as $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$.

b) solve equations in quadratic form.

7.3 a) Solution of Trigonometric Equations

Solve equations such as $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$.

(Note : $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$)

Example 1

Find the value of θ, if $0^0 \leq \theta \leq 360^0$

a) $\cos\theta = \frac{1}{2}$

b) $\sin\theta = -\frac{\sqrt{3}}{2}$

c) $\tan\theta = -2$

d) $\cos\theta = \sin 285^0$

Example 2

Solve the following trigonometric equations

a) $\tan 2\theta = \sqrt{3}$, $0° \leq \theta \leq 360°$

b) $\cos(3\theta - 75^0) = 0.5$, $0° \leq \theta \leq 360°$

7.3 b) Solve Equations In Quadratic Form

Example 3

Solve the following trigonometric equation

a) $2 \sin^2 x$ ï $\sin x - 1 = 0$ $\quad 0^0 \leq x \leq 360^0$

b) $3 \cot^2 \theta + 5 \operatorname{cosec} \theta + 1 = 0$ $\quad 0 \leq \theta \leq 2\pi$

Example 4

Show that the equation $2\sin x \cos x + 4\cos^2 x = 1$ can written in the form $\tan^2 x - 2\tan x - 3 = 0$. Hence, find the solution for x where $0 < x < 2\pi$.

LECTURE 4 OF 5

TOPIC : 7.0 TRIGONOMETRIC FUNCTIONS

SUBTOPIC : 7.3 Solution of Trigonometric Equations

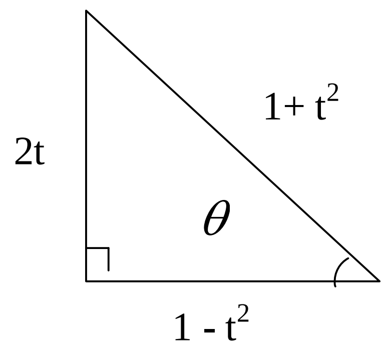
LEARNING OUTCOMES :

At the end of the lesson, students should be able to:

c) express sin θ, cos θ and tan θ in terms of t and solve $a\sin\theta \pm b\cos\theta = c$ where $t = \tan\frac{\theta}{2}$

c) Express sin θ, cos θ & tan θ in terms of t and solve $a\sin\theta \pm b\cos\theta = c$ where t = tan $\frac{\theta}{2}$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$



by using t = tan $\frac{\theta}{2}$

$$\tan\theta = \frac{2t}{1-t^2}$$

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

Example 1

If $\tan\frac{x}{2} = t$, express $\sin x$ and $\cos x$ in terms of t. Hence, find all values of t which satisfy $3\cos x - 4\sin x = 5$.

Example 2

If $\tan\frac{x}{2} = t$, find $\sin x$ and $\cos x$ in terms of t.
Hence, solve $\cos x + 7\sin x = 5$, for $0^0 \le x \le 180^0$.

Example 3

If $t = \tan\frac{\theta}{2}$, show that $\tan\theta = \frac{2t}{1-t^2}$.
Hence, solve the equation $3\cos\theta + 2\sin\theta - 3 = 0$ for $0 \le \theta \le 2\pi$.

Example 4

By using $t = \tan\frac{x}{2}$, prove that

a) $\dfrac{1}{\sin x} + \dfrac{1}{\tan x} = \dfrac{1}{\tan\frac{x}{2}}$

b) $\dfrac{1+\cos x}{1-\cos x} = \cot^2\dfrac{x}{2}$

LECTURE 5 OF 5

TOPIC : 7.0 TRIGONOMETRIC FUNCTIONS

SUBTOPIC : 7.3 Solution of Trigonometric Equations

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:

d) express $a\sin\theta \pm b\cos\theta$ in the form of $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \mp \alpha)$ and solve $a\sin\theta \pm b\cos\theta = c$ using $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \mp \alpha)$.

e) determine the maximum and minimum values of trigonometric expressions in the form of $a\sin\theta \pm b\cos\theta$.

7.3 d) Express $a\sin\theta \pm b\cos\theta$ in the form of $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \mp \alpha)$ and solve $a\sin\theta \pm b\cos\theta = c$ using $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \mp \alpha)$.

The equation 3 cos θ + sin θ =1 can also be solved if the expression 3 cos θ + sin θ is expressed in the form R cos (θ - α), where R > 0 and α are values to be determined.

Let R cos (θ - α) = a cos θ + b sin θ

R cos θ cos α + R sin θ sin α = a cos θ + b sin θ

Equating the coefficient of cos θ : R cos α = a é [1]

Equating the coefficient of sin θ : R sin α = b é [2]

$[1]^2 + [2]^2$: $R^2 \cos^2\alpha + R^2 \sin^2\alpha = a^2 + b^2$

$$R^2 (\cos^2\alpha + \sin^2\alpha) = a^2 + b^2$$

$$R^2 = a^2 + b^2$$

$$R = \sqrt{a^2 + b^2}$$

[2] ÷ [1] : $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{b}{a}$

$$\tan\alpha = \frac{b}{a}$$

Hence, the expression a cos θ + b sin θ can be expressed in the form R cos ($\theta - \alpha$), where $R = \sqrt{a^2 + b^2}$

and $\tan\alpha = \dfrac{b}{a}$

We conclude that :

$a\cos\theta \pm \mathrm{b}\sin\theta = \mathrm{R}\cos(\theta \mp \alpha)$ where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{\mathrm{b}}{\mathrm{a}}$
$a\sin\theta \pm \mathrm{b}\cos\theta = \mathrm{R}\sin(\theta \pm \alpha)$ where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{\mathrm{a}}{\mathrm{b}}$

Example 1

Express 3 cos θ - 4 sin θ in the form R cos (θ + α)

Example 2

Solve the following equations for

4 sin 2θ - 3 cos 2θ = 3 $0° \le \theta \le 360°$

Example 3

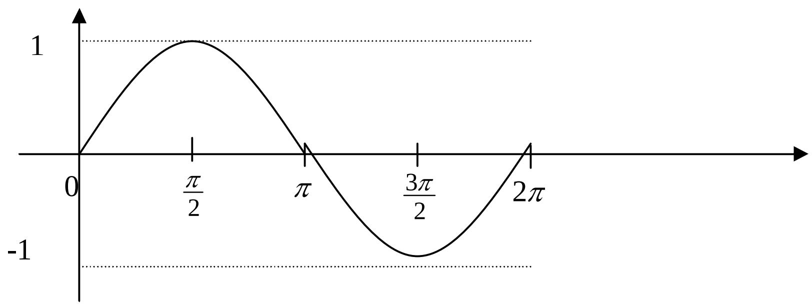
Solve the following equations for

2 cos θ + 3 sin θ =1 $0° \le \theta \le 360°$

7.3 e) Determine the maximum and minimum values of trigonometric expressions in the form of $a\sin\theta \pm b\cos\theta$.

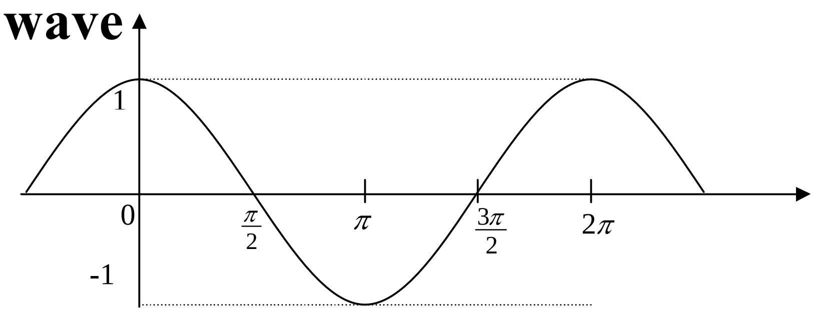
a sin x + b cos x can be rewritten using an expression of the form Rsin (x ± α) or R cos (x ± α) where α is the phase angle and R is the amplitude & depends on the values a and b.

sine wave



$$-1 \le \sin x \le 1$$

cosine wave



$$-1 \le \cos x \le 1$$

Maximum value

$\sin\frac{\pi}{2} = 1$

$\cos 0 = 1$ and $\cos 2\pi = 1$

Minimum value:

$\sin\frac{3\pi}{2} = -1$

$\cos\pi = -1$

Example 4

Find the maximum & minimum values for the expression 3 sin x ï 4 cos x and find the values of x , where $0^o < x < 360^o$, for which the expression has the extreme values.

Example 5

Show that $\dfrac{1}{3+\sqrt{5}} \leq \dfrac{1}{3+\sin\theta - 2\cos\theta} \leq \dfrac{1}{3-\sqrt{5}}$.