LECTURE 1 OF 5

TOPIC : 6.0 POLYNOMIALS

SUBTOPIC : 6.1 Polynomials
LEARNING OUTCOMES :

At the end of the lesson, students should be able to:
(a) perform addition, subtraction and multiplication of polynomials.
(b) perform division of polynomials and write the answer in the form
$P(x) = Q(x)D(x) + R(x)$, where the divisor can be linear or quadratic.

CONTENT

The Algebraic Operations on Polynomials

Algebraic operations can be performed on polynomials such as addition, subtraction, multiplication and division. The operations follow the commutative, associative and distributive laws of numbers.

Reference*:*
www.mathsisfun.com/algebra/polynomials.html

<u>Addition and subtraction</u>

The addition and subtraction of the polynomial $P(x)$ and $Q(x)$ can be performed by collecting like terms.

Example 1
Given $P(x) = 2x^4 - 5x^3 - 4$ and $Q(x) = x^4 + x^3 + 3x^2 + 4x$.
Determine
(a) $P(x) + Q(x)$ (b) $P(x) - Q(x)$

Reference :
math.about.com/library/blpoly.htm

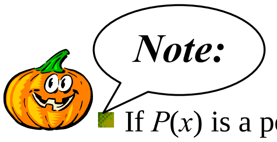
<u>Multiplication</u>

Note that every term in one polynomial is multiplied by each term in the other polynomial.

Example 2
Given $P(x) = x^2 - x - 1$ and $Q(x) = 2x^3 - x^2 + 1$.
Determine (a) $4Q(x)$ (b) $P(x)\,Q(x)$

Reference:
1. *www.mathsisfun.com/algebra/polynomials-multiplying.html*

2. *math.about.com/od/algebra1help/a/multiply-polynomials.htm*
3. *www.learnnc.org/lp/pages/2897*



- If $P(x)$ is a polynomial of degree m and $Q(x)$ is a polynomial of degree n, then product $P(x)Q(x)$ is a polynomial of degree $(m + n)$

Division

In the division of integer, $\frac{11}{2} = 5 + \frac{1}{2}$

- the quotient is 5
- the remainder is 1
- the divisor is 2

The statement could be expressed as

11 = 5(2) + 1 => divisor (quotient) + remainder

In the same way, the division of polynomials can be expressed in the form

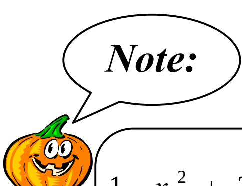
$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \quad \text{or} \quad P(x) = D(x)Q(x) + R(x)$$

When dividing polynomials, the quotient and remainder can be found by using long division.

$$\begin{array}{r|l} & Q(x) \\ \hline D(x) & P(x) \\ & \overline{R(x)} \end{array}$$

Example 3

Divide $x^2 + 2x - 5$ by x.



1. $x^2 + 2x - 5$ is called the dividend.
2. If the degree of $P(x)$ is n and the degree of $D(x)$ is r, then the degree of $Q(x)$ is (n ï r). The degree of $R(x)$ is less than or equal to (rï 1).

Example 4

Divide $2x^2 + 3x - 6$ by $x + 1$.

Example 5

Determine $\dfrac{3x^3 - 4x^2 + x + 7}{3x - 4}$ by long division.

Example 6

Divide $2x^4 + 8x^3 + 2x + 8$ by $2x + 8$.

Example 7

Divide $x^3 - 1$ by $x + 2$.

Example 8

Find the quotient and remainder for $\dfrac{x^3 - 2x + 5}{x^2 + 3x}$.

Example 9

Divide $7 - 6x^2 + 2x^4$ by $x^2 + 5$.

Example 10

Determine $\dfrac{x^4 + x^3 + x + 6}{(x+1)(x+3)}$.

Example 11

Determine $\dfrac{x^3 + 3x^2 + 8}{x^2 + 3x - 7}$.

Reference :

1. *http://www.purplemath.com/modules/polydiv2.htm*
2. *http://www.calc101.com/webMathematica/long-divide.jsp#topdoit*

LECTURE 2 OF 5

TOPIC : 6.0 POLYNOMIALS

SUBTOPIC : 6.2 Remainder Theorem, Factor Theorem and Zeroes of Polynomials.

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:
(a) apply the remainder and factor theorems
(b) identify the value of *a* such that (*x*+*a*) is a factor of *P*(*x*) and factorize *P*(*x*) completely.

CONTENT

The Remainder Theorem

When a polynomial *P*(*x*) is divided by a linear factor $x - a$, then the remainder is *P*(*a*)

Proof

Let *P*(*x*) be a polynomial of degree *n* where *n* Ó2. Then *P*(*x*) = *Q*(*x*) (*x* ï *a*) + *R(x)* (from polynomial division)

When *x* = *a*,
P(*a*) = *Q*(*a*) (*a* – *a*) + *R(a)*
Since (*a* ï *a*) = 0, then the remainder *R(a)* = *P*(*a*).

Note:
1. If *P*(*x*) is divided by *x*+*a* = $x - (-a)$, then *R*= $P(-a)$
2. If *P*(*x*) is divided by *ax*ï *b* = $a\left(x - \frac{b}{a}\right)$, then $R = P\left(\frac{b}{a}\right)$

Find the remainder when $P(x) = x^4 + x^3 - 2x^2 + 4x - 5$ is divided by:
(a) *x*+3 (b) 3*x*-1

Example 2

When $x^4 + kx^3 + 5x^2 - 6x - 8$ is divided by (*x* + 2) the remainder is 16. Determine *k*.

Example 3

The polynomial $ax^4 - 5x^3 + bx^2 - 7x + 1$ left a remainder of -8 when it is divided by (x-1) and a remainder of $\frac{11}{2}$ when divided by (2x+1), determine the values of a and b.

Example 4

Given that $P(x) = 2x^3 + ax^2 - 6x + 1$. When $P(x)$ is divided by $x + 2$, the remainder is twice of the remainder when $P(x)$ is divided by $x - 1$. Find a.

When a polynomial $P(x)$ is divided by a quadratic expression $ax^2 + bx + c$, then the remainder is $R(x)=ax+b$.

$$P(x) = Q(x).D(x) + R(x)$$
$$P(x) = Q(x).D(x) + (ax+b)$$

Example 5

Determine the remainder when $P(x) = 3x^4 + 5x^3 + x + 6$ is divided by $x^2 - 1$.

Example 6

When $x^4 + 4x^3 + px^2 + qx + 5$ is divided by $x^2 - 1$ the remainder is $2x + 3$. Find the values of p and q.

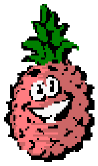
Reference :

1. *http://www.purplemath.com/modules/remaindr.htm*
2. *http://math.tutorvista.com/algebra/remainder-theorem.html*
3. *http://www.intmath.com/equations-of-higher-degree/2-factor-remainder-theorems.php*

The Factor Theorem

If the remainder obtained from dividing the polynomial $P(x)$ by $(x - a)$ is zero, then the linear term $(x - a)$ is called a factor of the polynomial $P(x)$.

If $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$



Note:

1. *Conversely, if (x-a) is a factor of P(x), then P(a)=0*
2. *In general, if (ax+b) is a factor of P(x), then* $P(-\frac{b}{a})=0$

Example 7

Determine whether the following linear functions are factors of the given polynomials:

a) $P(x) = 3 - 7x + 5x^2 - x^3, (x - 3)$
b) $P(x) = 2x^3 + 3x^2 - 8x + 3, (2x - 1)$
c) $P(x) = x^4 - 2x^3 + 3x - 6, (x + 1)$

Example 8
If $(2x + 1)$ is a factor of polynomial $P(x) = 2x^3 + px^2 - 5$, find the values of p.

Example 9
The polynomial $x^3 - 13x + k$ is exactly divisible by $(x - 4)$. Find the constant *k*.

Example 10

The polynomial $2x^3 + ax^2 + bx + 8$ has a factor $(x - 1)$ and gives a remainder of 50 when divided by $(x - 3)$. Determine *a* and *b*.

Example 11
Given that $(x + 2)$ is a factor of $P(x) = 2x^3 + x^2 + kx - 4$. Find the constant *k*. Hence, factorize the expression completely.

Example 12

Given that the expression $3x^3 + ax^2 + bx - 12$ is exactly divisible by $x^2 + 2x - 3$.
(a) Determine the values of *a* and *b*.
(b) Factorize the expression completely.

Reference :
http://www.mathsisfun.com/algebra/polynomials-remainder-factor.html

LECTURE 3 OF 5

TOPIC : 6.0 POLYNOMIALS

SUBTOPIC : 6.2 Remainder Theorem, Factor Theorem and Zeroes of Polynomials.

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:
(a) to find the roots of the equations and the zeroes of a polynomial

CONTENT

Roots and Zeros of a Polynomial

Definition

- A **zero** of a polynomial $P(x)$ is a number $\boldsymbol{a}$ such that $P(\boldsymbol{a}) = 0$
- $x = \boldsymbol{a}$ is called a **root** of the polynomial equation $P(x) = 0$

- In general, if $x = a$ is a root of a polynomial equation $P(x) = 0$ then $(x - a)$ is a factor of $P(x)$.
- Every polynomial equation of degree *n* has exactly *n* roots. Some of these roots may be repeated.

Example 1

Show that -4 is a zero of $6x^3 + 23x^2 - 5x - 4$.

Example 2

Factorize $P(x) = 2x^3 + 4x^2 - 22x - 24$ completely and write all the zeroes.

Example 3

Find the roots for the equation $x^3 + 3x^2 + x - 1 = 0$.

Example 4

Determine all the roots of $x^4 + x^3 - 7x^2 - x + 6 = 0$

Note:

- When attempting to factorize a polynomial of degree *4*, it is necessary to find two linear factors using the factor theorem.

LECTURE 4 OF 5

TOPIC : 6.0 POLYNOMIALS

SUBTOPIC : 6.3 Partial Fractions

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:

(a) define the partial fractions

(b) perform partial fractions decomposition when the denominators are in the form of
  (i) a linear factors, $(ax + b)$
  (ii) a repeated linear factor, $(ax + b)^n$

CONTENT

Types of fractions

- We call a fraction is a **proper fraction** when the degree of the numerator is less than the degree of the denominator.

  Example $\frac{3}{5x}$, $\frac{x^2}{x^3+1}$, $\frac{2x^2-7}{x^3+1}$

- Whereas a fraction is an **improper fraction** when the degree of the numerator is greater than, or equal to the degree of the denominator.

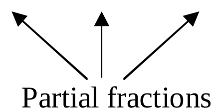
Example $\frac{5}{3}$, $\frac{2x^4 - 7}{x^3 + 1}$, $\frac{3x^3 + 7}{2x^3 + 5}$.

Definition

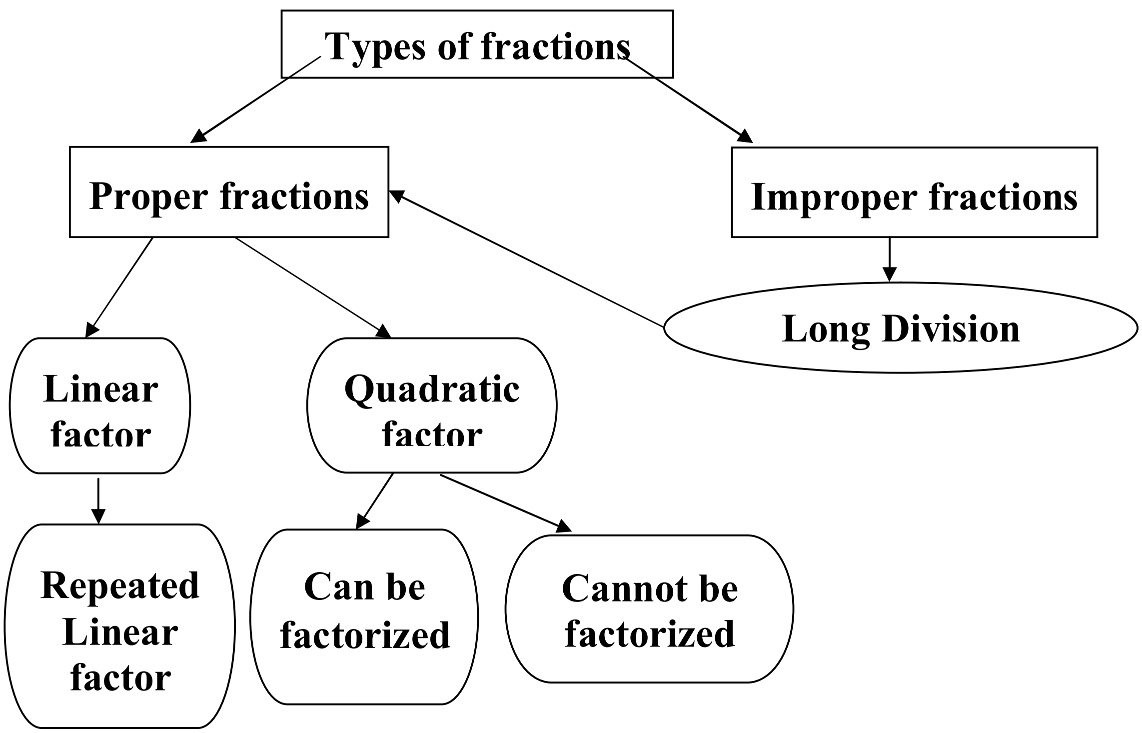
A fraction can be expressed or decomposed as the sum of two or more separate proper fraction.

This is known as **partial fractions**.

$$\frac{P(x)}{Q(x)} = \frac{a(x)}{b(x)} + \frac{c(x)}{d(x)} + \frac{e(x)}{f(x)}$$



Summary



PARTIAL FRACTIONS

Proper Fractions

Type 1: Denominator with distinct linear factors.

If $Q(x)$ is a product of linear factors, thus $\dfrac{P(x)}{(a_1x+b_1)(a_2x+b_2)\ldots\ldots(a_rx+b_r)}$ can be expressed as $\dfrac{A_1}{a_1x+b_1}+\dfrac{A_2}{a_2x+b_2}+\ldots\ldots+\dfrac{A_r}{a_rx+b_r}$ and $A_1, A_2, \ldots\ldots, A_r$ are constants.

Example

$$\frac{x+5}{(x-4)(x+6)}=\frac{A}{x-4}+\frac{B}{x+6} \quad \text{and} \quad \frac{7x-4}{(x+3)(x-5)}=\frac{A}{x+3}+\frac{B}{x-5}$$

Example 1

Express $\dfrac{2x-3}{(3x+2)(x+1)}$ in partial fractions.

Example 2

Express $\dfrac{5x+1}{x(x+1)(2x-1)}$ in partial fractions.

Type 2 : Repeated Factors

Denominator with repeated linear factors.

Example

The numerators are just constant terms

(a) $$\frac{6x^2-15x-8}{(x-3)(x+5)^2}=\frac{A}{x-3}+\frac{B}{(x+5)}+\frac{C}{(x+5)^2}.$$

Repeated factors

(b) $$\frac{x-1}{x^2(3x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3x+5}$$

Repeated factors

Example 3

Express $\dfrac{7x}{(x+2)^2}$ in partial fractions

Example 4

Express $\dfrac{2x+1}{x^2(x+2)}$ in partial fractions.

Example 5

Express $\dfrac{1}{x^2(x^2-1)}$ in partial fractions.

Reference :
http://www.mathsisfun.com/algebra/partial-fractions.html

LECTURE 5 OF 5

TOPIC : 6.0 POLYNOMIALS

SUBTOPIC : 6.3 Partial Fractions

LEARNING OUTCOMES :

At the end of the lesson, students should be able to:

(a) perform partial fractions decomposition when the denominators a quadratic factor, ($ax^2 + bx + c$) that cannot be factorize.

(b) transform the rational polynomials when the degree of the numerator is the same or more than that of the denominator into proper fraction and then determine the partial fractions.

CONTENT

PARTIAL FRACTIONS

Type 3: Denominator with quadratic factors.

Denominator with Quadratic factors that cannot be factorized

Example

(i) $$\frac{7x+10}{(x-2)(x^2+7x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+7x+3}$$

(ii) $$\frac{6x^2-8}{x(5x^2-6x+2)} = \frac{A}{x} + \frac{Bx+C}{5x^2-6x+2}$$

Example 1

Express the following as partial fractions:

(a) $\dfrac{3}{(x+2)(x^2+1)}$ (b) $\dfrac{4x+1}{(x+1)(x^2+5x+1)}$

Improper Fraction

If given that $\dfrac{P(x)}{Q(x)}$, where the degree of $P(x)$ Ódegree of $Q(x)$, then divide the denominator into the numerator to obtain the sum of a polynomial and a proper fraction.

Example 1

$$\frac{x^3+1}{x(x-1)} = (x+1) + \frac{(x+1)}{x(x-1)}$$

Polynomial   Proper Fraction

Example 2

Express $\dfrac{3x^2+5x-2}{x(x+3)}$ as a partial fraction.

Example 3

Express $\dfrac{x^3-x-5}{(x+2)(x^2+1)}$ as a partial fraction.

Example 4

Express $\dfrac{x^3}{x(x+1)^2}$ as a partial fraction.