LECTURE 1 OF 7

TOPIC : 5.0 FUNCTIONS AND GRAPHS

SUBTOPIC : 5.1 Functions

LEARNING OUTCOMES : At the end of the lesson, students are able to:

(a) define a function.
(b) use the vertical line test to determine whether a graph represent a function.
(c) use the algebraic approach or horizontal line test to determine whether a function is one-to-one.

CONTENT

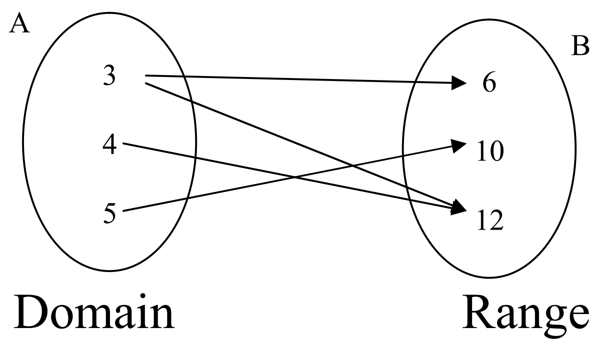
Definition

Relation

A relation is a correspondence between a first set, called the domain, and the second set, called the range, such that each member of the domain corresponds to at least one member of the range.

Example 1:

Let $A=\{3,4,5\}$ and $B=\{6,10,12\}$. Consider the relation “is a factor of”. This relation can be displayed using the arrow diagram as follows:



The relation can also written in the form of ordered pairs as $\{(3,6),(3,12),(4,12),(5,10)\}$.

Types of relation

- One to one
- One to many
- Many to one
- Many to many

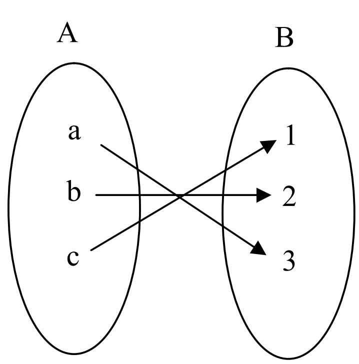
Function

A function is defined as a relation where every element in the domain has a unique image in the range.

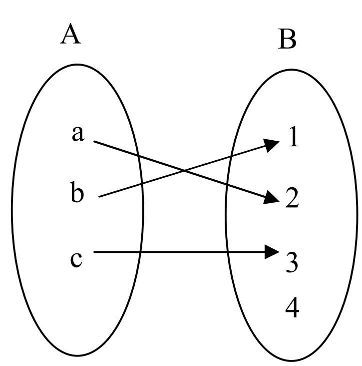
✓ In other words, a function are:

i) one to one relation. ii) Many to one relation.

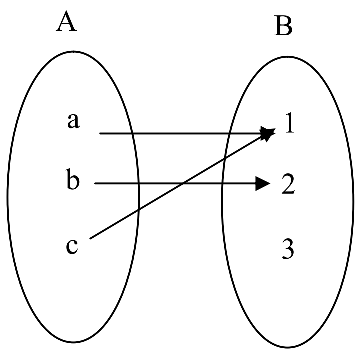
Examples of functions:



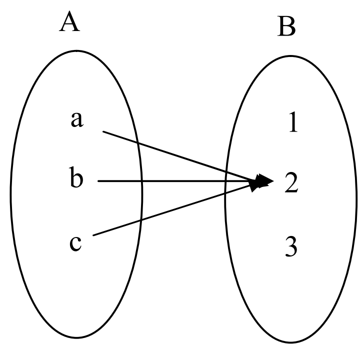
One-to-one relation and onto



One-to-one relation and not onto



Many to one relation and not onto



Many to one relation and not

✓ Mapping is another name for function.

✓ A mapping or function f from set A to set B is usually written as $f : A \to B$.

✓ If an element x, of set A is mapped into an element y in set B, so y is an image of x.

✓ The image of x is thus represented by $f(x)$ and we write $y = f(x)$

<u>Example 2:</u>

Let A = {1, 2, 3, 4} and B = {set of integers}. Illustrate the function $f : x \to x + 3$

Solution:

The graph of a function

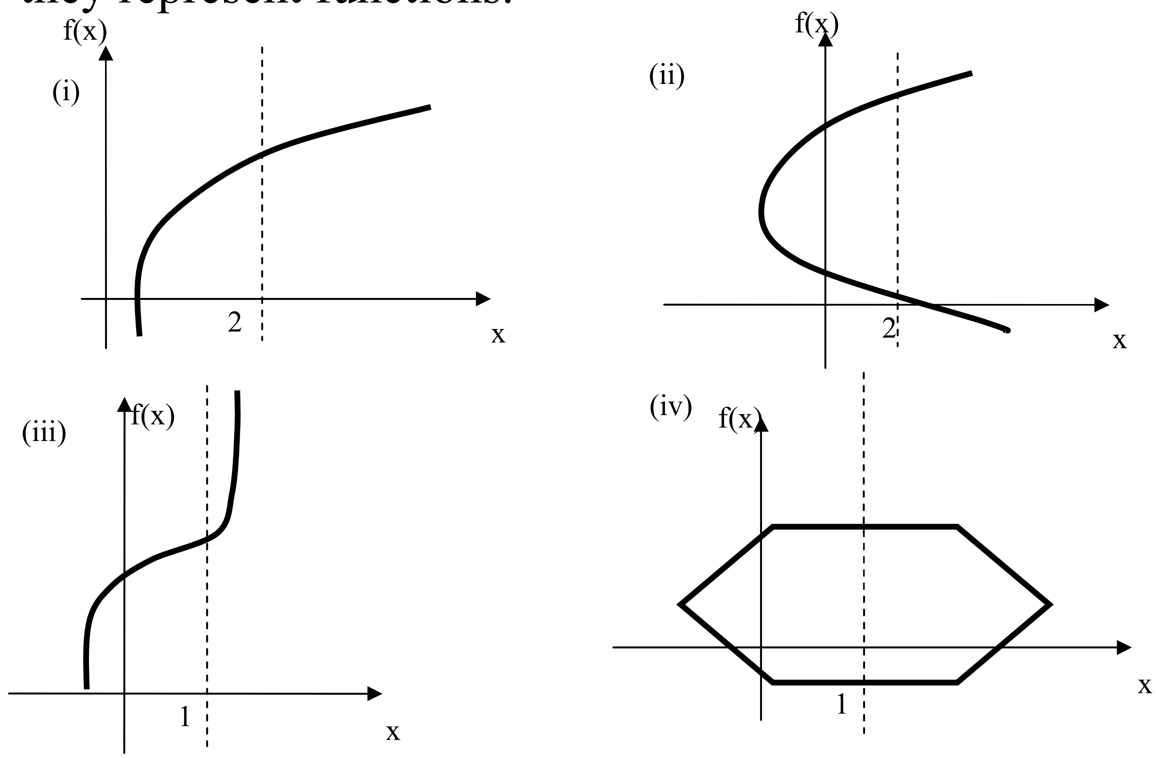
- most common method of representing function is by a graph
- each graph is drawn with the coordinate axes
- horizontal axis (x – axis) representing the domain
- vertical axis (y – axis) representing the range.

Vertical line Test

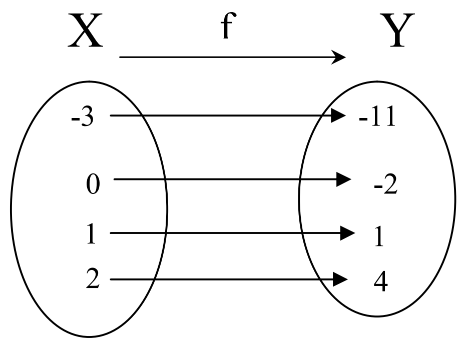
The vertical line test is a graphical method use to determine whether a relation in *x* is a function. If any vertical line drawn intersects the curve $y = f(x)$ only at one point, then $f(x)$ is a function of *x*.

Example 3:

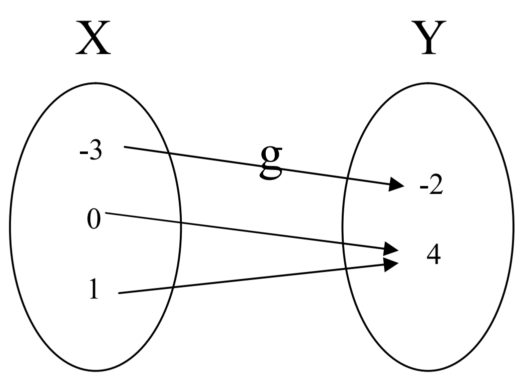
Consider the graphs shown below and state whether they represent functions.



One –to one functions



In the above arrow diagram, every element of set X is mapped to exactly one element of set Y. Function *f* which has this property is known as a **one-to-one function**.



In the above arrow diagram, two elements, namely 0 and 1, of set X are mapped to the same element of 4 of set Y, that is g(0)=g(1)=4. As such, function g is **not a one-to-one function.**

There are two methods to determine whether a function is one-to-one:

(a) **Horizontal Line test (Graphically Method)**
The horizontal line test is a graphical method used to determine is a function is one to one. In general, if any horizontal line drawn intersects the graph of the function only at one point, then the function is one-to-one function.

(b) **Algebraic Method**
A function *f* with a domain *X* is called a one-to-one function if two elements of X have the same image, that is $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$. To prove that a function is one-to-one, we must show that

$f(x_1) = f(x_2)$ implies that $x_1 = x_2$

Example 4:
Use the horizontal line test (graphical method) to determine whether each of the following functions is one-to-one function.

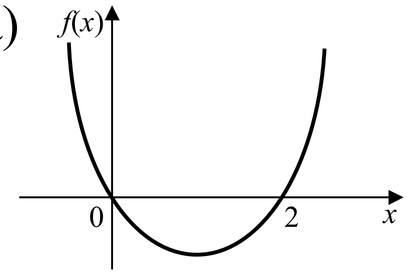
(a) $f(x) = x(x-2)$ (b) $f(x) = x^3 - 1$

(c) $f(x) = \sqrt{x+1}$ (d) $f(x) = \dfrac{-2}{x+1}$

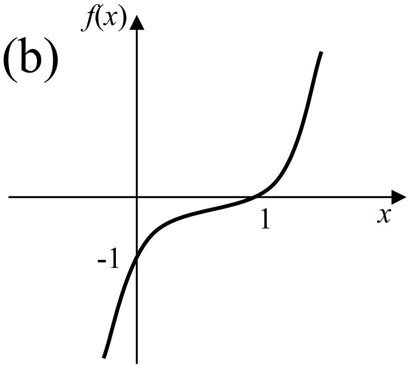
(e) $f(x) = |x-1|$

Solution:

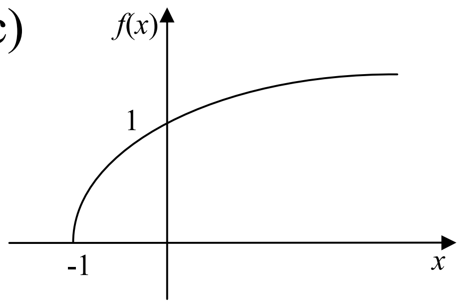
(a)



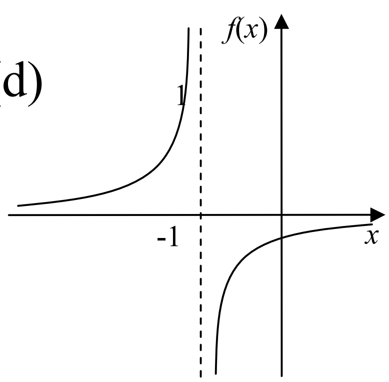
(b)



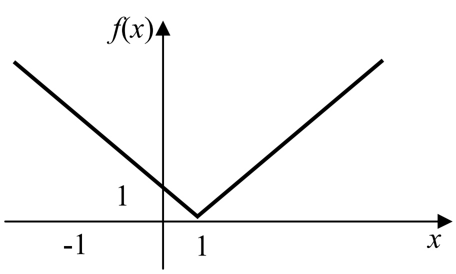
(c)



(d)



(e)



Example 5:

By using the algebraic method, determine whether *f* is a one to one function or not

(a) $f(x) = 2x + 3$ (b) $f(x) = x^2 + 2x - 5$

(c) $f(x) = \dfrac{2}{x+3}$

(d) $f(x) = \sqrt{x+4}$ (e) $f(x) = |x - 3|$

Solution:

LECTURE 2 OF 7

TOPIC : 5.0 FUNCTIONS AND GRAPHS

SUBTOPIC : 5.1 Functions

LEARNING OUTCOMES : At the end of the lesson, students are able to:

(d) sketch the graph of a function

(e) state the domain and range of a function

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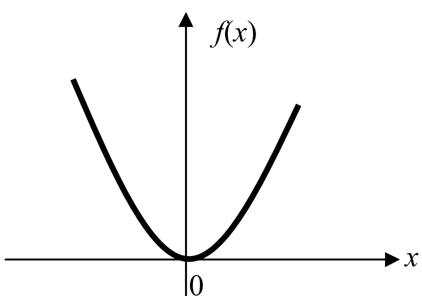
CONTENT

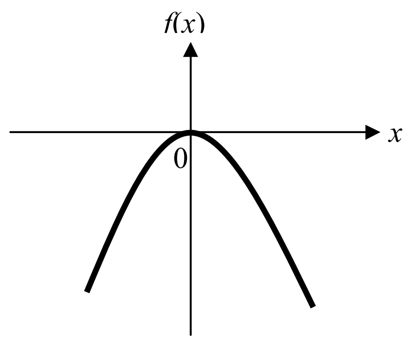
Basic shape of a function

(i) Quadratic function

(a) $f(x) = x^2$

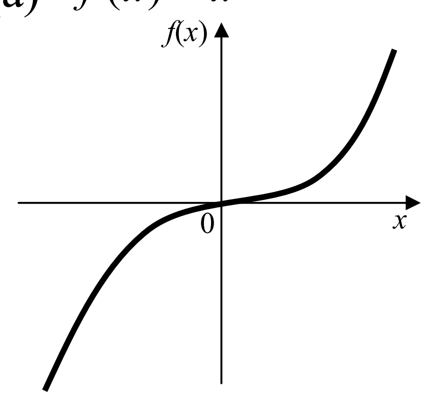


(b) $f(x) = -x^2$

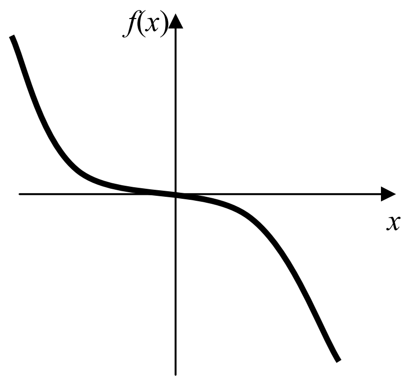


(ii) Cubic function

(a) $f(x) = x^3$



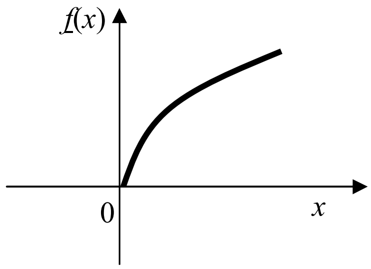
(b) $f(x) = -x^3$



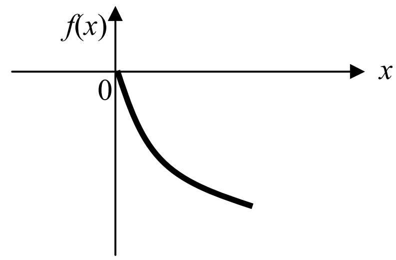
(iii) surd function

The graph exist only for $x \geq 0$

(a) $f(x) = \sqrt{x}$

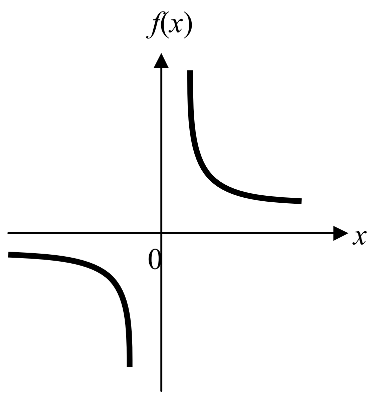


(b) $f(x) = -\sqrt{x}$

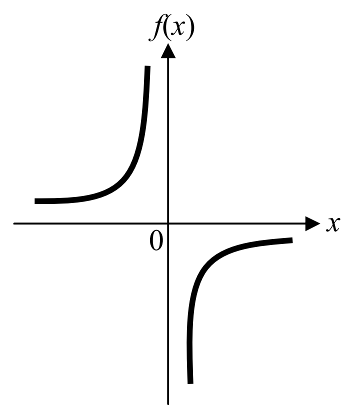


(iv) Reciprocal function

(a) $f(x) = \dfrac{1}{x}$

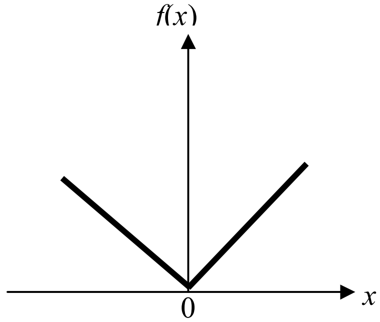


(b) $f(x) = -\dfrac{1}{x}$



(v) Absolute value function, |f(x)|

$f(x) = |x|$



Example 1:

Sketch the graph of the following functions.

(a) $f(x) = -5$

(b) $f(x) = -x + 2$

(c) $f(x) = x^2 - x - 2$

(d) $f(x) = x^2(2 - x)$

(e) $f(x) = \sqrt{x+5}$

(f) $f(x) = |x - 2|$

(g) $f(x) = \dfrac{1}{x-2}$

(h) $f(x) = \begin{cases} -x^2, & x < 0 \\ x + 5, & x \geq 0 \end{cases}$

Solution:

Domain and Range

Given $y = f(x)$

- Domain, D_f, is the set of the values of *x* in which $f(x)$ is defined.
- Range , R_f, is the set of all possible value of $f(x)$ as *x* varies throughout the domain . R_f is a collection of all image of *f*.
- Domain and range of function can be written in the form of sets or interval notations.
- There are two methods to find the domain and range of a function $f(x)$
  (i) Graphically
  (ii) Algebraic

> The domain and the range of the function can be determined by means of graph, the horizontal axis representing the domain and the vertical axis, the range.

Example 2:

Sketch the graph of the following functions. Hence, find its domain and range.

(a) $f(x) = 2 - 2x$

(b) $f(x) = x^2 - 4x - 5$

(c) $f(x) = -x^3 + 8$

(d) $f(x) = \sqrt{x - 3}$

(e) $f(x) = -\dfrac{1}{2x - 5}$

(f) $f(x) = |3x - 1|$

(g) $f(x) = \begin{cases} -x + 2, & -1 \le x \le 1 \\ 3, & x = 1 \\ x, & x > 1 \end{cases}$

Solution:

LECTURE 3 OF 7

TOPIC : 5.0 FUNCTIONS AND GRAPHS

SUBTOPIC : 5.2 Composite Functions

LEARNING OUTCOMES : At the end of the lesson, students are able to:

(a) represent composite function by an arrow diagram
(b) find composite functions.
(c) find one of the functions when the composite and the other function are given.

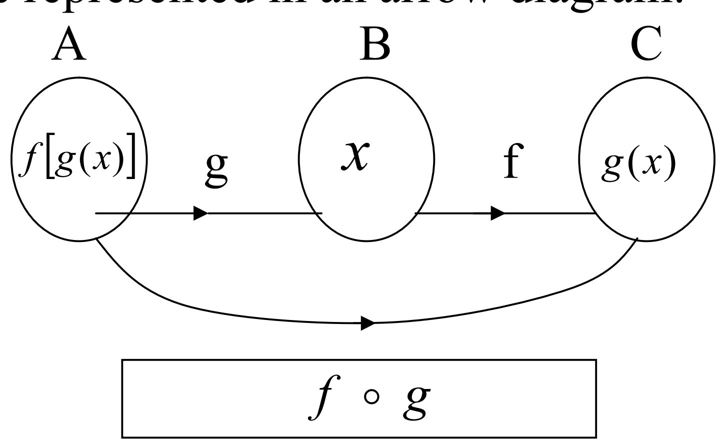
CONTENT

Definition:

Consider two functions $f(x)$ and $g(x)$.

We define $f \circ g(x) = f[g(x)]$ meaning that the output values of the function g are used as the input values for the function f.

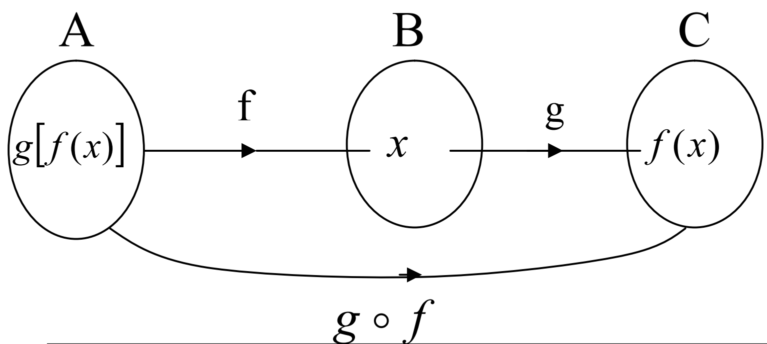
This can be represented in an arrow diagram:



Note that (f o g)(x) ≠ f(x) g(x).

Similarly, we define $g \circ f(x) = g[f(x)]$ meaning that the output values of the function f are used as the input values for the function g.

This can be represented in an arrow diagram.



Note that (g o f)(x) ≠ g(x) f(x).

<u>Example 1:</u>

If $f(x) = 3x + 1$ and $g(x) = 2 - x$, find as a function of x

(a) $f \circ g$ (b) $g \circ f$

Solution:

Note that (f o g)(x) ≠ (g o f)(x).

Example 2:

The function f and g are defined by $f : x \rightarrow 3x^2 + 1$ and $g : x \rightarrow 5x - 7$, find:

(a) $fg(x)$ (b) $ff(x)$ (c) $gg(x)$

Solution:

Example 3:

If $f(x) = 2x - 1$ and $g(x) = x^3$, find the values of :

(a) $gf(3)$ (b) $fg(3)$ (c) $f^2(3)$

Solution:

Note that $f^2(3) \neq [f(3)]^2$

Example 4:

Given that $f(x) = 2x$, $g(x) = 1 + x$ and $h(x) = x^2$, find the functions:

(a) $fgh(x)$ (b) $hgf(x)$ (c) $ghf(x)$

Solution:

Example 5:

The functions *f*, *g* and *h* are defined by $f(x) = 2 - x$, $g(x) = \frac{3}{x+1}$ and $h(x) = 2x - 1$

(a) Show that $f^2(x) = x$.

(b) Find an expression for $g^2(x)$,

(c) Solve the equation $h^3(x) = x$.

Solution:

Example 6:

Given that $g(x) = x^2 + 1$ and $gf(x) = x^2 + 4x + 5$, find the function of $f(x)$.

Solution:

Example 7:

If $g(x) = 3 + x$ and $fg(x) = x^2 + 6x + 10$, find the function of $f(x)$.

Solution:

Example 8:

The function *f* and *g* are defined by $f(x) = x + 4$, $g(x) = x^2$ respectively. Find the function of *h* such that $hgf(x) = x^2 + 8x + 3$

Solution :

Example 9:

If $fg(x) = 4x^2 - 2x + 1$ and $g(x) = 2x + 1$, find the function of $gf(x)$. Subsequently, find the values of x that satisfy $fg(x) = gf(x)$.

Solution:

LECTURE 4 OF 7

TOPIC : 5.0 FUNCTIONS AND GRAPHS

SUBTOPIC : 5.3 Inverse Functions

LEARNING OUTCOMES : At the end of the lesson, students are able to:

(a) determine the inverse of a function.
(b) determine whether a function has an inverse and find the inverse of a function.

CONTENT

The Inverse Of A Function

Fig. 1 shows the mapping of the domain {-3, 0, 1, 2} by the function $f(x) = 3x - 2$.
Verify that the range is {-11, -2, 1, 4}.

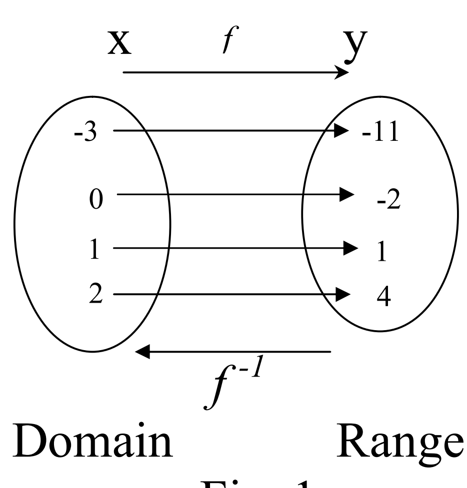


Fig. 1

Is there a function that will map back to the domain? The function $f(x)$ in Fig. 1 mapped *x* onto *y* where $y = 3x - 2$. Now we wish to start with *y* and return to *x*.

If $3x - 2 = y$. So, $x = \frac{y + 2}{3}$.

So this new function will map *y* onto $\frac{y+2}{3}$

Testing this with $y = -11$, we get $\frac{-11+2}{3} = -3$ which is the original value of *x*.

We can check with other values. Such a function, *if it exists,* is called the inverse function of *f* and is written as f^{-1}. (Read this as 'inverse *f*').
Usually we take x as the 'starting' letter so we have
$f^{-1}(x) = \frac{x+2}{3}$.

For the function $f(x) = 3x - 2$, its inverse $f^{-1}(x) = \frac{x + 2}{3}$.

The inverse of a function f exists if and only if f one-to one function.

If $f : x \to y$ is a function, then $y \to x$ is also a function. Thus, the inverse function of *f* can be written as $f^{-1} : y \to x$. To verify that f^{-1} is the inverse of *f*, show that

$$f\left[f^{-1}(x)\right] = x \quad \text{or} \quad f^{-1}\left[f(x)\right] = x$$

Example 1:

Show whether the following functions are one-to-one. For functions that are one-to-one, find their inverse functions.

(a) $f(x) = 3x + 2$

(b) $g(x) = x^2 + 4x + 1$

(c) $p(x) = -x^2 + 5, \quad x \geq 0$

(d) $q(x) = \dfrac{2}{x}$

(e) $k(x) = |x + 3|$

LECTURE 5 OF 7

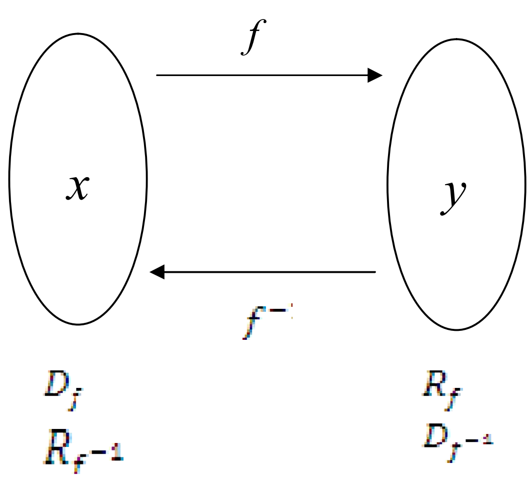
TOPIC : 5.0 FUNCTIONS AND GRAPHS

SUBTOPIC : 5.3 Inverse Functions

LEARNING OUTCOMES : At the end of the lesson, students are able to:
(c) Identify the domain and range of an inverse function.

CONTENT

Domain and Range of Inverse function

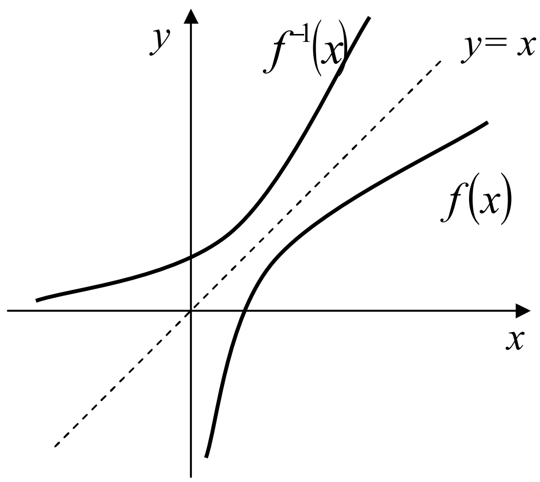


From the diagram :
Domain of $f(x)$ = Range of $f^{-1}(x)$
Range of $f(x)$ = Domain of $f^{-1}(x)$

Graph of Inverse Function

With the property of the inverse function $D_f = R_{f^{-1}}$ and $R_f = D_{f^{-1}}$, that means point (x, y) of $f(x)$ is changed to (y, x) of $f^{-1}(x)$ and achieved by reflecting the points about $y = x$.

Graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.



Example 1:

Find the inverse of $f(x) = \frac{1}{1-x} + 2, x \neq 1$ and state the domain of the inverse function.

Solution:

Example 2:

Function f and g are defined as $f(x)=\frac{2x-5}{x+3}$ and $g(x)=\frac{3x+5}{2-x}$.

(a) Find $fg(x)$ and deduce $f^{-1}(x)$

(b) Determine the domain and range of $f^{-1}(x)$

Solution:

Example 3:

The functions f and g are defined by $f(x)=2x+3$ and $g(x)=x-1$. Find

(a) f^{-1} and g^{-1}

(b) $gf^{-1}(x)$ and $fg^{-1}(x)$

(c) $(fg)^{-1}(x)$

(d) $f^{-1}g^{-1}(x)$

Solution :

Example 4:

Given that $f(x) = (x-1)^2 + 2$ for $x \geq 1$. Find the $f^{-1}(x)$ and state its domain and range.

Hence, sketch the graph of $f(x)$ and $f^{-1}(x)$ on the same axis.

Solution :

LECTURE 6 OF 7

TOPIC : 5.0 FUNCTIONS AND GRAPHS
SUBTOPIC : 5.4 Exponential and Logarithmic Functions

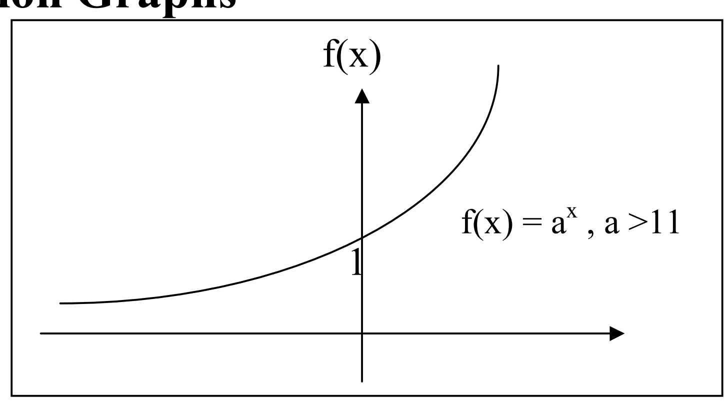
LEARNING OUTCOMES : At the end of the lesson, students are able to:
(a) determine the relationship of exponential and logarithmic functions graphically and algebraically.
(b) find the domain and range of an exponential and logarithmic functions.

CONTENT

Exponential function is $f(x) = a^x$ where $x \in R$, $a > 0$ and $a \neq 1$. Constant *a* is known as the base and variable *x* is known as the exponent.
Important class of exponential function is one where the base is given by Euler's number, *e*. Euler's number, *e* is an irrational number where $= \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$.
Value of *e* is approximately 2.718281828

Basic Exponential Function Graphs

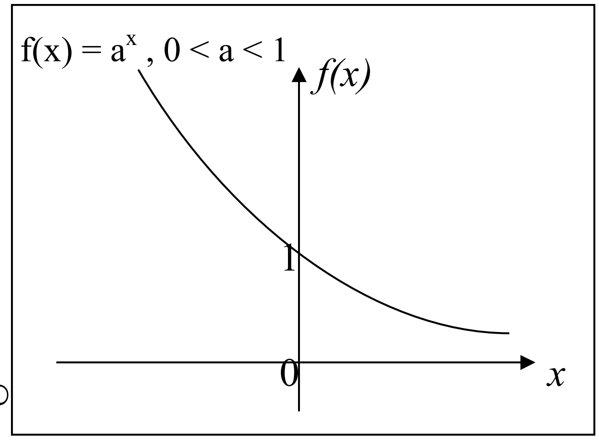
(a) $f(x) = a^x$, $x \in$R, a > 1
Basic properties:
i) *f(x)* > 0 for *x*∈R.
ii) When *x* = 0, *f(x)*=1



iii) When $x \to \infty$, $f(x) \to \infty$

iv) When $x \to -\infty$, $f(x) \to 0$

(b) $f(x) = a^x$, $x \in R$, $0 < a < 1$



Basic properties:

i) $f(x) > 0$ for $x \in R$.

ii) When $x = 0$, $f(x) = 1$

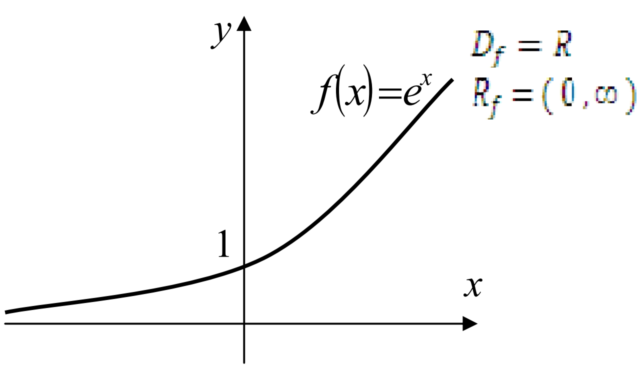
iii) When $x \to \infty$, $f(x) \to 0$

iv) When $x \to -\infty$, $f(x) \to \infty$

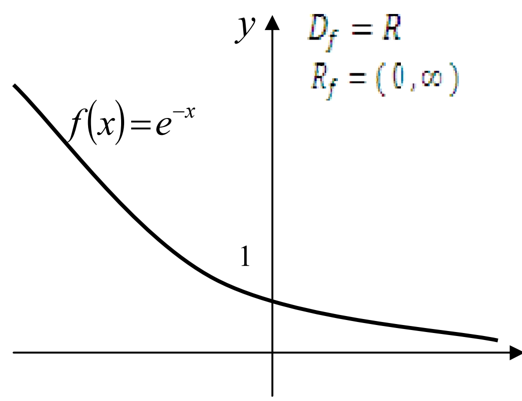
Example 1:

Sketch the graph of:

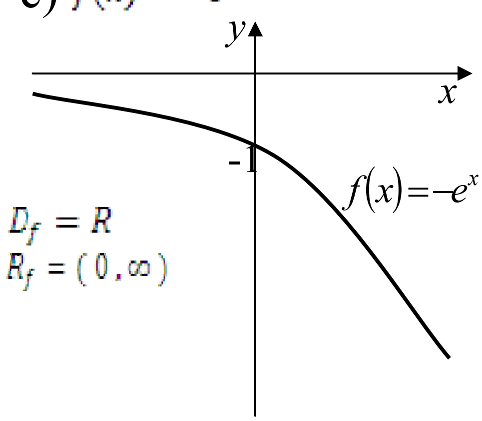
a) $f(x) = e^x$



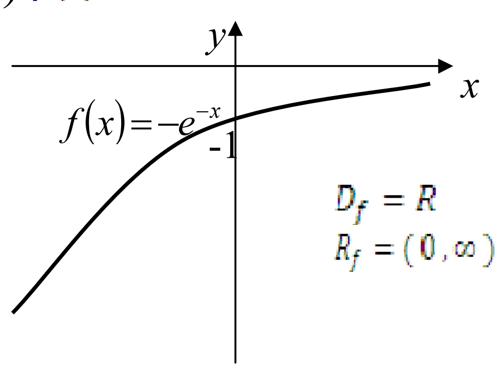
b) $f(x) = e^{-x}$



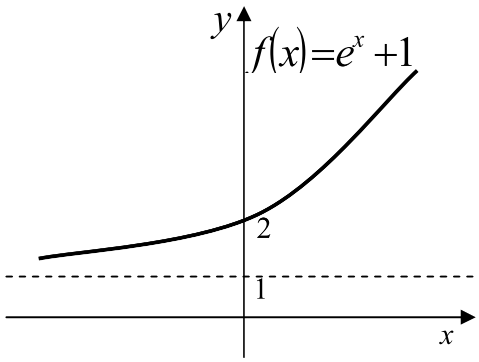
c) $f(x) = -e^x$



d) $f(x) = -e^{-x}$

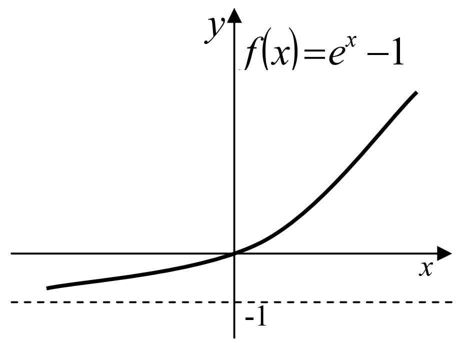


e) $f(x) = e^x + 1$



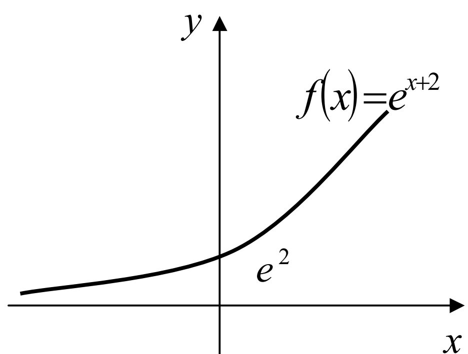
$D_f = R$
$R_f = (1, \infty)$

(f) $f(x) = e^x - 1$



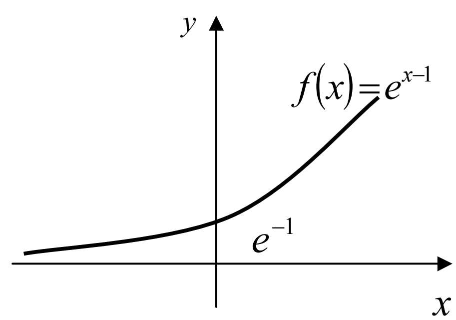
$D_f = R$
$R_f = (-1, \infty)$

(g) $f(x) = e^{x+2}$



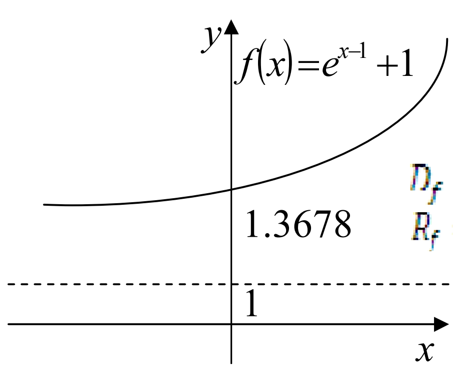
$D_f = R$
$R_f = (0, \infty)$

(h) $f(x) = e^{x-1}$



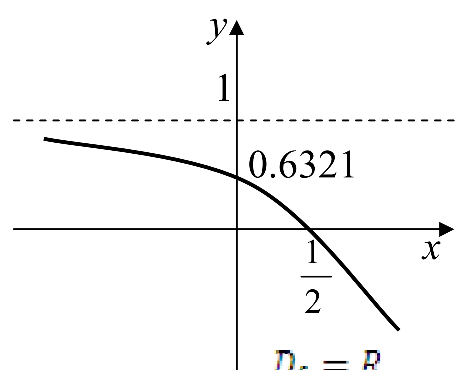
$D_f = R$
$R_f = (0, \infty)$

(i) $f(x) = 1 + e^{x-1}$



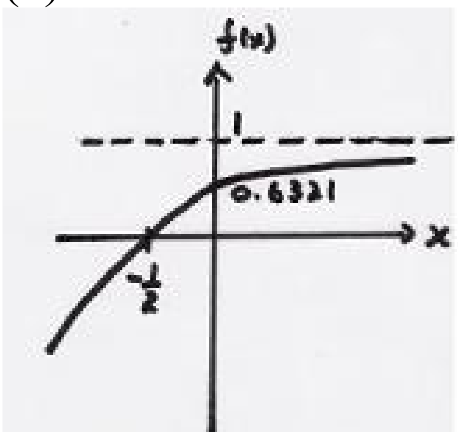
$D_f = R$
$R_f = (1, \infty)$

(j) $f(x) = 1 - e^{2x-1}$



$D_f = R$
$R_f = (-\infty, 1)$

(k) $f(x) = 1 - e^{-2x-1}$



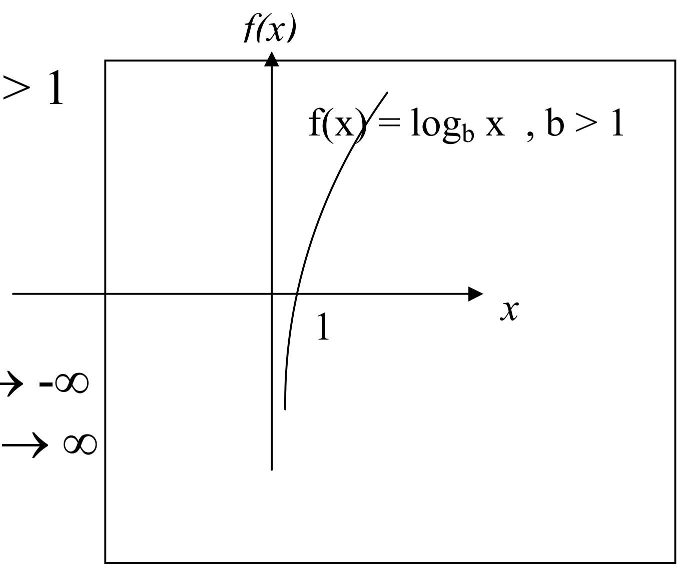
$D_f = R$
$R_f = (-\infty, 1)$

Logarithmic Function

A logarithmic function is a function of the form $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$. The constant a is known as the base and the variable x is any positive real number.

Basic Logarithmic Function Graphs

(a) $f(x) = \log_b x \quad , x \in R , b > 1$



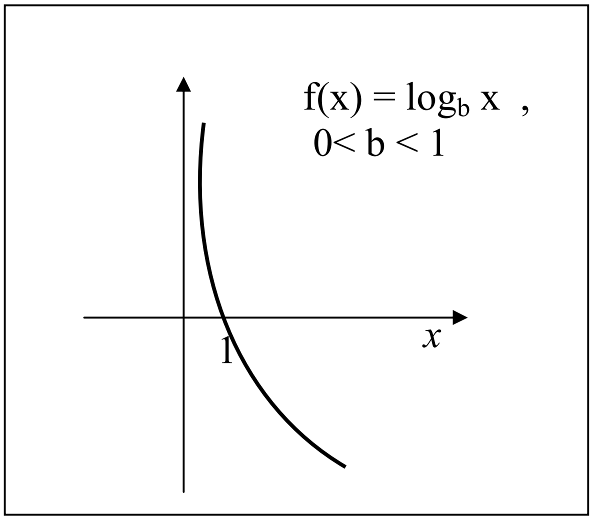
Basic properties :

i) When $x = 1$, $f(x) = 0$
ii) When $x \to 0$, $f(x) \to -\infty$
iii) When $x \to \infty$, $f(x) \to \infty$

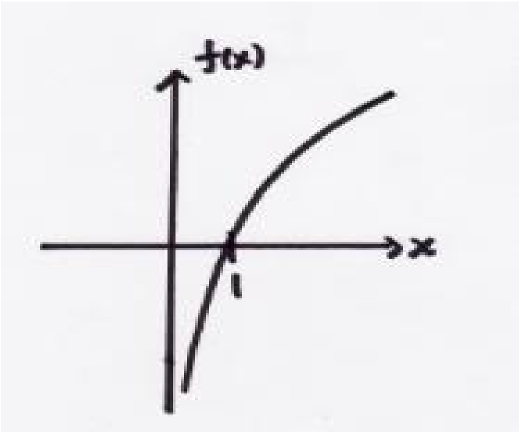
(b) $f(x) = \log_b x \quad , x \in R$, $0 < b < 1$

Basic properties :

i) When $x = 1$, $f(x) = 0$
ii) When $x \to 0$, $f(x) \to \infty$
iii) When $x \to \infty$, $f(x) \to -\infty$

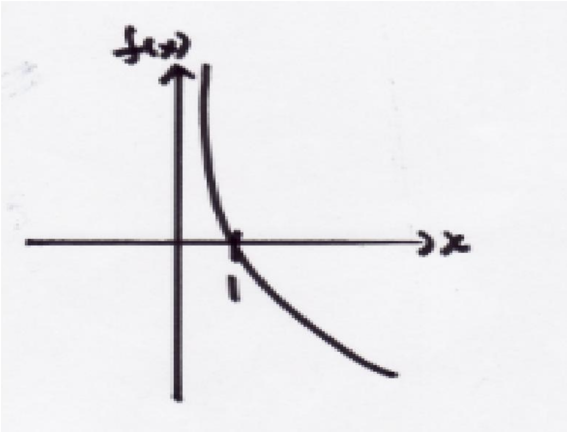


Example 2:
Sketch the graph of:
(a) $f(x) = \ln x$



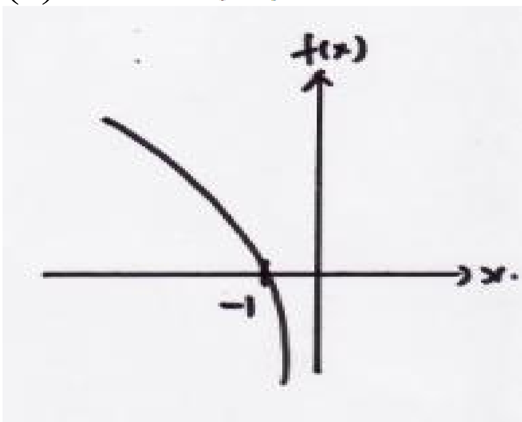
$D_f = (0,\infty)$, $R_f = R$

(b) $f(x) = -\ln x$



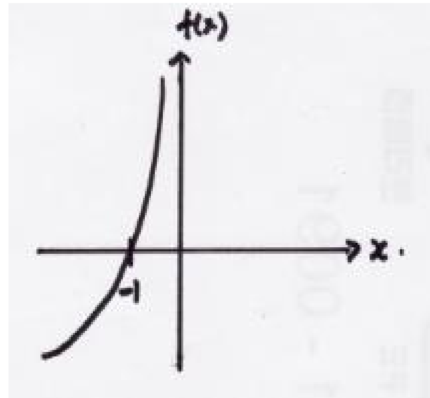
$D_f = (0,\infty)$, $R_f = R$

(c) $f(x) = \ln(-x)$



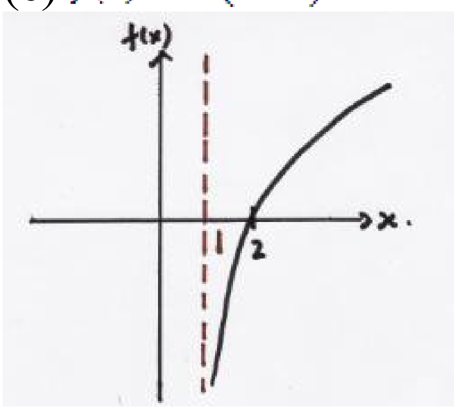
$D_f = (-\infty, 0)$, $R_f = R$

(d) $f(x) = -\ln(-x)$



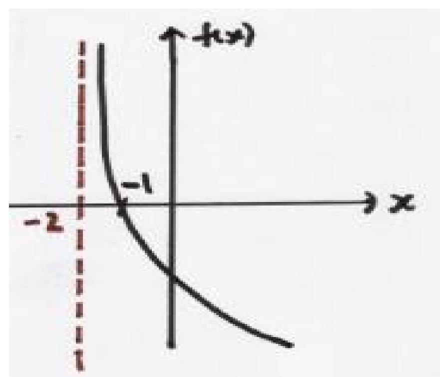
$D_f = (-\infty, 0)$, $R_f = R$

(e) $f(x) = \ln(x-1)$



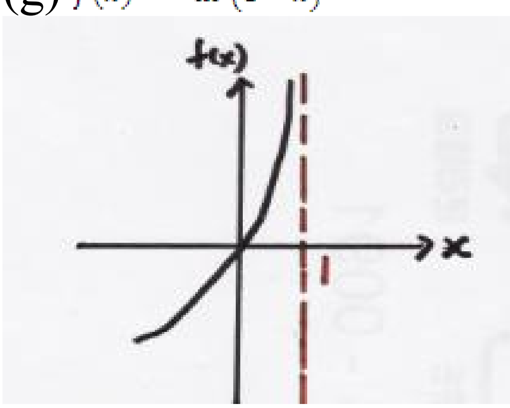
$D_f = (1, \infty)$, $R_f = R$

(f) $f(x) = -\ln(x+2)$



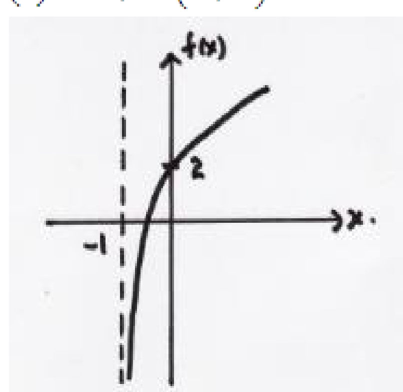
$D_f = (-2, \infty)$ $R_f = R$

(g) $f(x) = -\ln(1-x)$



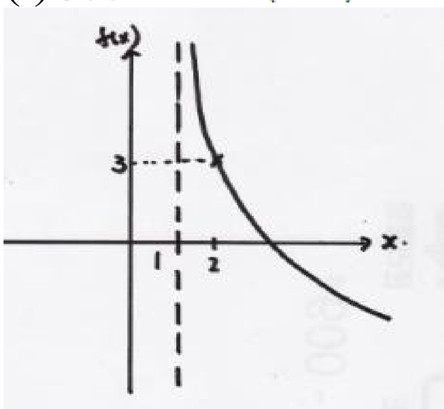
$D_f = (-\infty, 1)$, $R_f = R$

(h) $f(x) = 2 + \ln(x+1)$



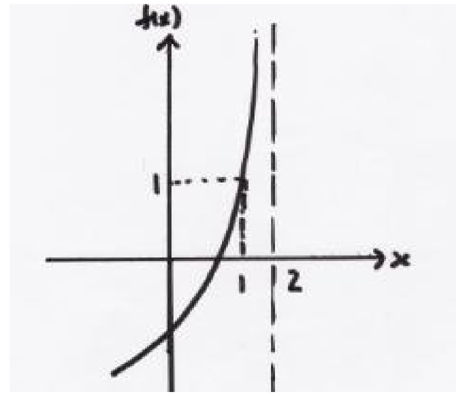
$D_f = (\;\; 1, \infty)$, $R_f = R$

(i) $f(x) = 3 - 2\ln(x-1)$



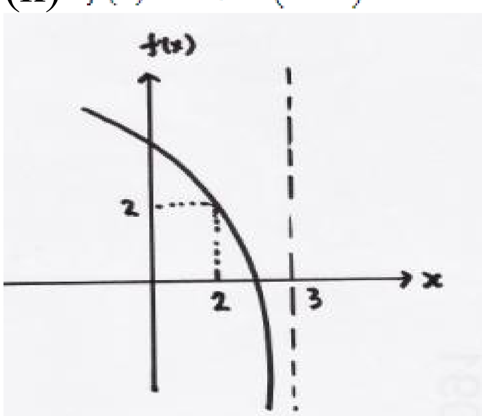
$D_f = (1,\infty)$
$R_f = R$

(j) $f(x) = 1 - \ln(2-x)$



$D_f = (-\infty,2)$
$R_f = R$

(k) $f(x) = 2 + \ln(3-x)$



$D_f = (-\infty,3)$
$R_f = R$

Relationship Between An Exponential and Logarithmic Function

Exponential function

The inverse of exponential function, $f(x) = a^x$ is

Let $f[f^{-1}(x)] = x$

$$a^{f^{-1}(x)} = x$$

$$f^{-1}(x) = \log_a x$$

The inverse of exponential function is a logarithmic function.

Logarithmic function

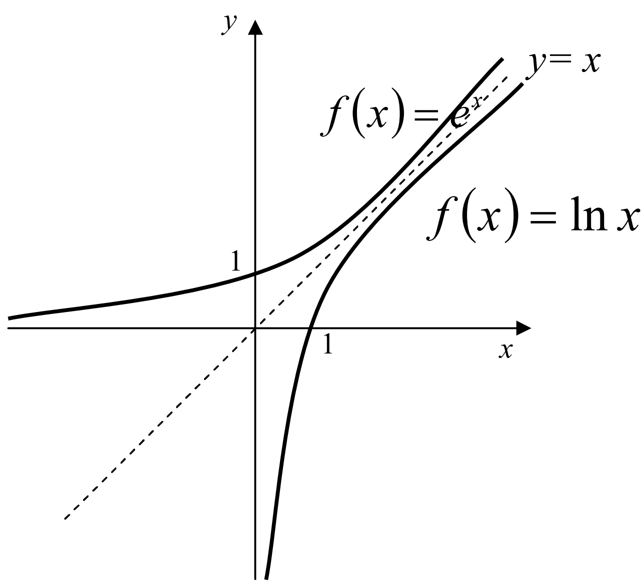
The inverse of logarithmic function, $f(x) = \log_a x$ is

Let $f[f^{-1}(x)] = x$

$\log_a f^{-1}(x) = x$

$f^{-1}(x) = a^x$

The inverse of logarithmic function is exponential function.



LECTURE 7 OF 7

TOPIC : 5.0 FUNCTIONS AND GRAPHS

SUBTOPIC : 5.4 Exponential and Logarithmic Functions

LEARNING OUTCOMES : At the end of the lesson, students are able to:

(c) determine the composite functions involving exponential and logarithmic functions.

(d) Sketch the graph involving exponential and logarithmic functions.

Example 1:

Given that $f(x) = 2x + 1$ and $g(x) = e^{x}$, find the $gf(x)$ and $fg(x)$..

Solution:

Example 2:

Functions *f* and *g* are defined as $f(x) = \ln(x-2)$ and $g(x) = 2x + 3$.

(a) Find $fg^{-1}(x)$ and $g^{-1}f(x)$

(b) Sketch the graph of $fg^{-1}(x)$ and $g^{-1}f(x)$

Solution:

Example 3:

Given $f(x) = lnx$, find $f^{-1}(x)$ and sketch the graph of $f(x)$ and the $f^{-1}(x)$ on the same axes.

Solution:

Example 4:

Given $f(x) = \ln(3x+2)$, show that f is one-to-one function and

(a) find the $f^{-1}(x)$,

(b) sketch the graph of $f(x)$ and the $f^{-1}(x)$ on the same axes.

Solution:

Example 5:

Function f is given by $f(x) = 5 + 2e^{-2x}$.

(a) Use algebraic method to show that f is a one-to-one function.

(b) Determine $f^{-1}(x)$. State the domain and range of $f^{-1}(x)$.

(c) Show the relationship between the graphs of f and $f^{-1}(x)$ on the same diagram.

Solution :