LECTURE 1 OF 4

TOPIC : 3.0 SEQUENCES AND SERIES

SUBTOPIC : 3.1 Sequence And Series

LEARNING OUTCOMES : At the end of the lesson, students should be able to:

(a) write n^{th} term of simple sequences and series

(b) Determine the *n*th term of arithmetic sequence and series, $T_n = a + (n-1)d$ and to use the sum formula, $S_n = \frac{n}{2}[2a + (n-1)d]$ and $S_n = \frac{n}{2}(a + l)$

CONTENT

Definition Of Sequence And Series

A sequence is a function *f* whose domain is the set of natural numbers. The value $f(1), f(2), f(3), ...$ are called the terms of the sequence.

A series is the sum of the terms of a sequence. We write the sum of the first *n* terms of a sequence as S_n, where

$$S_n = T_1 + T_2 + ... + T_n$$

Simple examples of a sequence:

(a) $1,3,5,7,...$
(b) $2,4,6,8,...$

Solution

(a) In this case, 1, 3, 5, 7,… $2n+1,2n+2,2n+3,...$
$T_n = \{2n-1\}_{n=1}$ consider $n=1$.

(b) In this case, $2,4,6,8,..2n,2n+2,...$
$T_n = \{2n\}_{n=1}$

Example 1

Find the first five terms of the sequence defined by each formula.

(a) $T_n = 2n^2 - 1$
(b) $T_n = n^2 - n$
(c) $T_n = \dfrac{n}{n+1}$

Solution

Example 2

Find the *n*th term of a sequence whose first several terms are given.

(a) $4, 7, 10, 13, 16$

(b) $-2, 4, -8, 16, -32$

(c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$

Solution

We notice that the numerators of these fractions are the odd numbers $\{2n-1\}$ and the denominators are the even numbers $\{2n\}$,

$$\therefore T_n = \left\{\frac{2n-1}{2n}\right\}_{n=1}^{\infty}$$

Tips

Odd numbers; $1,3,5,7,...2n-1$ (if $n=1$)

$1,3,5,7,...2n+1$ (if $n=0$)

$1,3,5,7,...2n+3$ (if $n=-1$)

Even numbers; 2,4,6…$2n$ (if n=1)

2,4,6,…$2n$+2 (if n =0)

The Partial Sums Of A Sequence (Series)

For the sequence, $T_1, T_2, T_3, ...T_n, ...$

The partial sums are,

$$S_1 = T_1$$
$$S_2 = T_1 + T_2$$
$$S_3 = T_1 + T_2 + T_3$$
$$S_n = T_1 + T_2 + T_3 + ... + T_n \implies \boxed{T_n = S_n - S_{n-1}}$$

S_1 is called the first partial sum.

S_2 is called the second partial sum and so on.

$S_1, S_2, S_3, \ldots S_n$ are called sequences of partial sums.

Note

1. $T_1 + T_2 + T_3 + T_4 + T_5$ is a finite series

2. $T_1 + T_2 + T_3 + T_4 + T_5 + \ldots$ is an infinite series

Example 3

Find the T_4 and T_5, $S_4 + S_5$ for the sequence given by 1,4,7,...

Solution

Definition Arithmetic Sequences

An arithmetic sequences is a sequence of the form $a, a+d, a+2d, a+3d, \ldots$. The number a is the first term and d is the common difference of the sequence. The nth term of an arithmetic sequence is given by.

$$T_n = a + (n-1)d$$

Proof, $T_1 = a$

$$T_2 = a + d$$

$$T_3 = a + 2d$$

$$\vdots$$

$$T_n = a + (n-1)d$$

Example 4

Find T_n term of the arithmetic sequence below.

$$9, 4, -1, -6, -11, \ldots$$

Solution

Example 5

Find the first six terms and the 300th term of the arithmetic sequence.

$13, 7, 1, \ldots$

Solution

Example 6

The 11th term of an arithmetic sequence is 52 and the 19th term is 92, find the 1000th term.

Solution

Partial sums of arithmetic sequence

From $S_n = T_1 + T_2 + T_3 + \ldots T_n$, since $T_1 = a, T_2 = a + d$ and so on,
Thus,

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+(n-1)d) \quad \text{...(1)}$$

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \ldots + (a+2d) + (a+d) + a \quad \text{.....(2)}$$

(1) + (2), obtained

$$2S_n = n[2a + (n-1)d]$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Notice that $T_n = a + (n-1)d$, so we can write

$$S_n = \frac{n}{2}[a + a + (n-1)d]$$

$$= \frac{n}{2}[a + l] \quad \text{since } T_n = a + (n-1)d$$

Example 7

Find the sum of the first 50 odd numbers.

Solution

Example 8

How many terms of the arithmetic sequence $5,7,9,...$ must be added to get 572.

Solution

Example 9

An insurance representative made RM 30 000 her first year and RM 60 000 her seventh year.
Assume that her annual salary figures form an arithmetic sequence and predict what she will make in her tenth year.

Solution

LECTURE 2 OF 4

TOPIC : 3.0 SEQUENCES AND SERIES

SUBTOPIC : 3.1 Sequences and Series

LEARNING OUTCOMES : At the end of the lesson, students should be able to:

(a) determine the *n*th term, $T_n = ar^{n-1}$ and to use the sum formula $S_n = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$.

Definition

A Geometric Progression (GP) is a sequence of numbers in which any term can be obtained from the previous term by multiplying by a certain number called the common ratio.

The *n*th term: $T_n = ar^{n-1}$

The terms of a GS are given by $a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$ where a is the first term and r is the common ratio.

First term : $T_1 = a$

Second term : $T_2 = ar$

Third term : $T_3 = ar^2$

Fourth term : $T_4 = ar^3$

⋮

*n*th term : $T_n = ar^{n-1}$

Thus, the general term of a geometric sequence is given by $T_n = ar^{n-1}$

The common ratio, r is given by $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = ... = \frac{T_n}{T_{n-1}}$

Example 1

Find the general term for the geometric sequence 8, 16, 32, 64,…

Solution

Example 2

Find the 9th term of the geometric sequence 27, 9, 3, …

Solution

Example 3

The *n*th term of a sequence $T_n = \left(\frac{2}{3}\right)^{n-1}$

(a) Show that the sequence is a geometric progression .

(b) Find the value of *n* where $T_n = \frac{256}{6561}$

Solution

The Sum of Geometric Sequence (Progression).

Writing the sum of the first n terms of a GS as S_n, it follows that

$$S_n = a + ar + ar^2 + ... + ar^{n-2}, ar^{n-1}$$

multiplying by *r* gives

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$$

subtracting

$$S_n - rS_n = a - ar^n$$

or $\quad S_n(1-r) = a(1-r^n)$

if $r \neq 1$, $\quad S_n = \dfrac{a(1-r^n)}{(1-r)}$

Example 4

Find S_5 of the GP in which a =27 and $r = \dfrac{2}{3}$

Solution

Example 5

The first term of a GP is 27 and its common ratio is $\frac{4}{3}$. Find the least number of terms the sequence can have if its sum exceeds 550.

Solution

Sum To Infinity of a Geometric Sequence

The sum to infinity of a GS in which -1 < r <1 is given by,

$$s_\infty = \frac{a}{1-r}$$

Example 6

Calculate the sum to infinity of the series $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$

Solution

Example 7

a) What is the sum to infinity of the GS with the first term is 500 and common ratio $\frac{2}{5}$?

b) The sum to infinity of a GP is 90. If the common ratio is $\frac{3}{5}$, what is the first term?

Solution

Example 8

The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Show that the common ratio, r, satisfies the equation $1-49r^2 = 0$.
Hence find the first term of the GP with positive common ratio.

Solution

Example 9

Express each of this number as a quotient $\frac{a}{b}$

(a) $1.\overline{6}$ (b) $5.4\overline{59}$

Solution

Example 10

Each year the price of a car depreciates by 10% of the value at the beginning of the year. If the original price of the article was RM 60 000, find its price after 12 years.

Solution

LECTURE 3 OF 4

TOPIC : 3.0 SEQUENCES AND SERIES

SUBTOPIC : 3.2 Binomial Expansion

LEARNING OUTCOMES : At the end of the lesson, students should be able to:

(a) Expand $(a+b)^n$ where *n* is a positive integer.

(b) Write *n!* notations and ${}^nC_r = \binom{n}{r}$ as a binomial coefficient.

(c) Determine the general term in a binomial expansion $(a+b)^n$ where *n* is a positive integer.

Expansion of $(a+b)^n$

We know that

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

What about $(a+b)^6$?

By using a Pascal Triangle

						1						
					1		**1**					
				1		**2**		**1**				
			1		**3**		**3**		**1**			
		1		**4**		**6**		**4**		**1**		
	1		**5**		**10**		**10**		**5**		**1**	
1		**6**		**15**		**20**		**15**		**6**		**1**

$$(a+b)^6 = a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$(a+b)^6$ also can be expanded by using the binomial expansion.

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0b^n$$

For any positive integer n

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0b^n$$

Example 1

By using binomial expansion, expand the following:

(a) $(2+x)^5$ (b) $(1-2x)^4$

Solution

Definition of n factorial

If n is a positive integer, then n! is given by

$$n! = n(n-1)(n-2)(n-3)\ldots 1$$

Example

(a) $5! = 5 \times 4 \times 3 \times 2 \times 1$

$= 120$

(b) $9! = 9 \times 8 \times 7 \times 6 \ldots \times 1$

$= 362{,}880$

Definition of a binomial coefficient

Let n and r be whole numbers with n$\geq$ r, then

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

Example 2

(a) $\binom{5}{2} = {}^{5}C_{2}$ (b) $\binom{9}{3}$

Solution

The $(r+1)$ term of the binomial expansion of $(a+b)^n$ is $\binom{n}{r}a^{n-r}b^r$

For the binomial expansion of $(a+b)^8$

The 3rd term $= \binom{8}{2}a^6b^2$

$= 28a^6b^2$

And the 6 th term $= \binom{8}{5}a^3b^5$

$= 56a^3b^5$

Example 3

Find the coefficients of y^5 and x^6 in the expansion of $(x+y)^9$

Solution

Example 4

Determine the terms independent of x in the following binomial expansions

a) $\left(x + \dfrac{1}{x} \right)^8$

b) $\left(x^2 - \dfrac{3}{x^2} \right)^{10}$

Solution

LECTURE 4 OF 4

TOPIC : 3.0 SEQUENCES AND SERIES

SUBTOPIC : 3.2 Binomial Expansion

LEARNING OUTCOMES : At the end of the lesson, students should be able to:

(d) Expand $(1+x)^n$ for $|x|<1$ where n is a rational number.

CONTENT

For the binomial expansion $(1+x)^n$ where n is the negative integer or a fraction, we use the following expansion.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

This expansion is valid for $|x|<1$, i.e, $-1 < x < 1$

Example 1

By using the binomial expansion, solve the following:

(a) $(1+x)^{-2}$

(b) $(1+x)^{\frac{1}{2}}$

(c) $(8-x)^{\frac{1}{3}}$

(d) $(2+3x)^{-1}$

Solution

Example 2

Find the first 4 terms of the expansion of $(1+x)^6$ and $(1-x)^6$. Hence find the value of $(1.001)^6$ and $(0.999)^6$

Solution

Example 3

Expand $(1+3x)^{\frac{1}{3}}$ in ascending powers of x until and including the term x^3. By substituting $x=\frac{1}{125}$, find the value of $\sqrt[3]{2}$ to 5 decimal places.

Solution

Exercise

a) Expand $(1+x)^{\frac{1}{2}}$ and hence find the value of $\sqrt{1.08}$ to 4 decimal places.

b) Expand $(1-x)^{\frac{1}{2}}$ in ascending powers of x until and including the term x^3. By taking $x=\frac{1}{64}$, find the value of $\sqrt{7}$ to 3 decimal places.

(Ans : a) 1.1.0392 b) 2. 2.646)