LECTURE 1 OF 6

TOPIC:
2.0 Equations, Inequalities and Absolute Values

SUBTOPIC:
2.1 Equations

LEARNING OUTCOMES:
At the end of the lesson, student should be able to solve equations which involve indices, surds and logarithms.

CONTENT

An algebraic expression contains one or more terms that are combined through basic operations such as addition, subtraction, multiplication or division. Some examples are:

$$2x+5,\ \frac{3x-1}{1+x},\ 5^x+1,\ 3-\sqrt{x},\ 2\log_4(1-x)+10.$$

An equation is a mathematical statement that states two algebraic expressions are equal.

Some examples in one variable are:

$2x+5=0,\quad \dfrac{3x-1}{1+x}=4,\quad 5^x+1=5x,\quad 3-\sqrt{x}=\sqrt{x+1},$

$2\log_4(1-x)+\log_2 x=10.$

To solve an equation in one variable means to find the values of that variable that make the equation true. These values are called solutions or roots of the equation.

Index Equations

Three methods of solving index equations:

(1) By comparing base and index of the terms on the left and on the right.

(2) By taking log of both sides of the equation.

(3) By substitution to get a quadratic equation.

Example 1

Solve the equations:

(a) $5^x=125$ (b) $7^{x+1}=12$ (c) $x^{\frac{2}{3}}=100$

(d) $27^{x+1}=9^{x-1}$ (e) $343^{x^2}=\dfrac{1}{7^{1-5x}}$

Example 2

Solve the following equations:

(a) $5^{2x+1} = 6(5^{x}) - 1$

(b) $4^{x+1} - 5(2^{x}) + 1 = 0$

(c) $e^{2x} - 3e^{x} + 2 = 0$

Surd Equations

Solving surds equation, normally requires the need to square both sides of the equation and remember to check the answer.

Example 3

Solve each of the following equation :

(a) $\sqrt{2x-1}-5=0$

(b) $\sqrt{3x+1}+1=x$

(c) $\sqrt{x}+\sqrt{x+2}=2$

(d) $\sqrt{3x+1}-\sqrt{2x-1}=\sqrt{x+2}$

LECTURE 2 OF 6

TOPIC:
2.0 Equations, Inequalities and Absolute Values

SUBTOPIC:
2.1 Equations

LEARNING OUTCOMES:
At the end of the lesson, student should be able to solve equations involving logarithms.

CONTENT

Logarithmic Equations

Solving equations involving logarithms normally requires the changing of the equations in the form of logarithms with same base.

Example 1

Solve the equation $2^{x-1} = 3^{x}$

Example 2

Solve the equation $\log_3 x - 4 \log_x 3 + 3 = 0$

Example 3

Solve the equation $\log_2 x + \log_2 (x - 7) = 3$

Example 4

Solve the equation $3 \ln 2x - 4 = 2 \ln 2x$

Example 5

Solve $\log_2 (\log_3 x) = 4$

Exercise

1. Express y in terms of x for each of the following equations

   a) $\log y = 2 \log x$ ($y = x^2$)

   b) $1 + \log y = 3 \log x$ ($y = \frac{x^3}{10}$)

2. Find the value of x if

   a) $7^x = 8$ ($x = 1.0686$)

   b) $5^{2x} = 8$ ($x = 0.6460$)

   c) $3^{x+1} = 4^{x-1}$ ($x = 8.6336$)

4. Solve the equation $\log_4 x + \log_x 4 = 2.5$ (2 or 16)

LECTURE 3 OF 6

TOPIC:
2.0 Equations, Inequalities and Absolute Values

SUBTOPIC:
2.2 Inequalities

LEARNING OUTCOMES:

At the end of the lesson, the students should be able to:

(a) Relate the properties of inequalities.
(b) Solve linear inequalities.
(c) Solve quadratic inequalities by using graphical approach

CONTENT

The Properties of Inequalities

If $a > b$ then

i) $a + c > b + c$
ii) $ac > bc$, $c > 0$
iii) $ac < bc$, $c < 0$
iv) $\frac{1}{a} < \frac{1}{b}$ and $a, b \neq 0$

Example 1

Give the solution set for the following inequalities.

(a) $2x - 4 > 8$

(b) $4x - 5 < 2x + 9$

(c) $3x + 5 \geq x - 7$

Example 2

Solve the following inequalities and write the answer in the interval form.

(a) $-7 < 2 + 3x < 8$

(b) $-2 \leq \dfrac{-x+4}{2} \leq 9$

(c) $6 < 4 - x \leq 12$

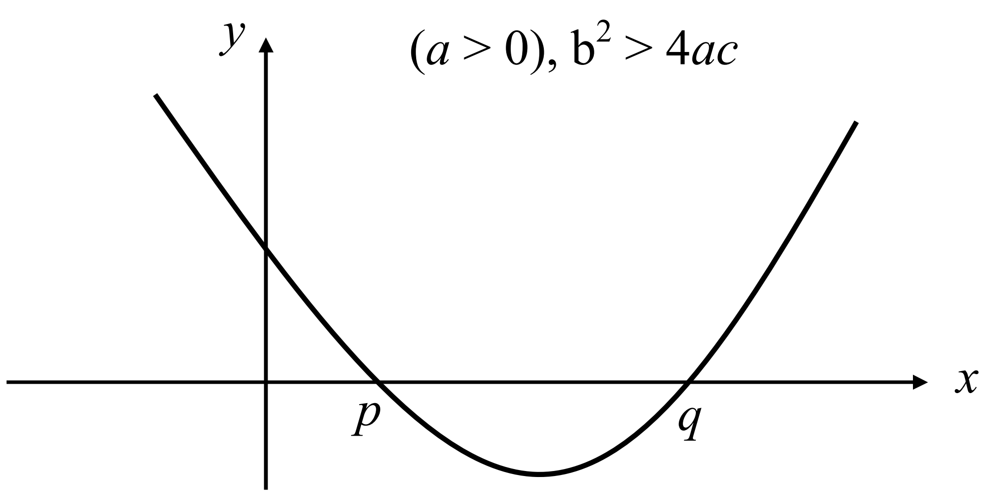
Quadratic Inequalities

A quadratic inequality is an inequality of the form $ax^2 + bx + c > 0$ where a, b and c are real number with $a \neq 0$. The inequality symbols $<$, $\leq$ and $\geq$ may also be used.

Quadratic inequalities can be solved by using graphical or algebraic approach.

Graphical Approach:

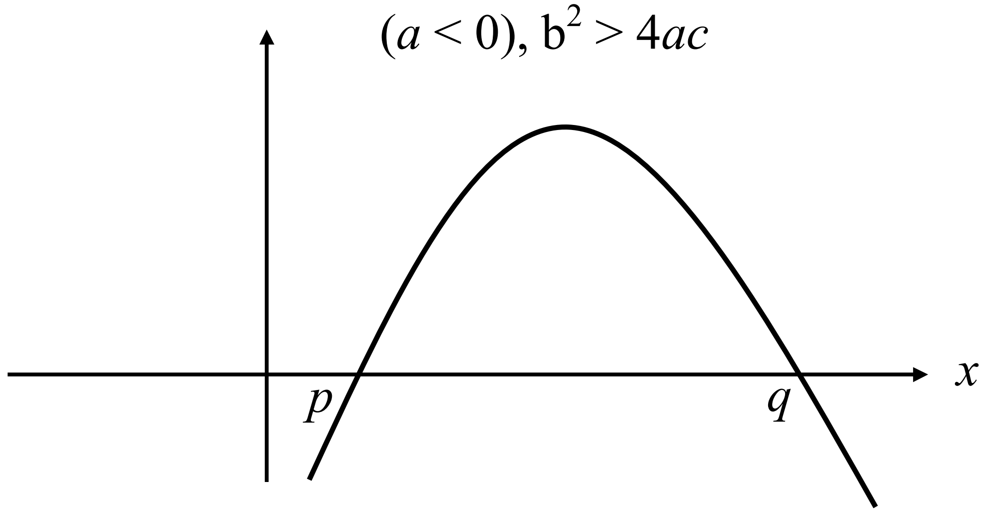
The graph of quadratic expression $y = ax^2 + bx + c$ is sketched and points where the graph cuts the x-axis, say p and q are noted.



$y > 0$ when $x < p$ or $x > q$
$y < 0$ when $p < x < q$

y

$(a < 0)$, $b^2 > 4ac$



$y > 0$ when $p < x < q$

$y < 0$ when $x < p$ or $x > q$

Example 3

Solve the following inequalities by using graphical approach.

(a) $x^2 + 6x + 5 > 0$

(b) $6 - 7x - 3x^2 < 0$

(c) $3x(x-5) \le 2(2x-3)$

Exercise

1. Solve the following linear inequalities

   (a) $6x - 5 > x + 20$

   (b) $-2 \leq \dfrac{7-3x}{4} < 10$

   Answer:

   (a) $x > 5$ (b) (-11, 5]]

2. Solve the following quadratic inequalities

   (a) $x^2 + 2x - 15 < 0$

   (b) $4 + 3x - x^2 \leq 0$

   (c) $2x^2 + x - 10 \geq 0$

   Answer:

   (a) (-5, 3)

   (b) $(-\infty, -1) \cup [4, \infty)$

   (c) $(-\infty, -\dfrac{5}{2}) \cup [\,2, \infty)$]

LECTURE 4 OF 6

TOPIC:
2.0 Equations, Inequalities and Absolute Values

SUBTOPIC:
2.2 Inequalities

LEARNING OUTCOMES:

At the end of the lesson, student are able to:

(a) Solve quadratic inequalities by using algebraic approach

(b) Solve rational inequalities involving linear expressions.

CONTENT

Algebraic Approach:

Theorem:

1. $ab > 0$ if and only if
   $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$

2. $ab < 0$ if and only if
   $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$

Example 1

Solve the following quadratic inequalities by using algebraic approach

(a) $x^2 - 2x - 15 \geq 0$

(b) $(2x-1)(x+3) < 4x$

Rational Inequalities

Rational inequalities are inequalities that can be expressed in the form :

$$\frac{P(X)}{Q(X)} > 0, \ \frac{P(X)}{Q(X)} \geq 0, \frac{P(X)}{Q(X)} < 0, \text{and } \frac{P(X)}{Q(X)} \leq 0;$$

$Q(X) \neq 0$, where $P(X)$ and $Q(X)$ are linear expression.

Example: $\dfrac{2x-1}{x+3} \leq 0$ and $\dfrac{3x+1}{x-5} \geq 2$

Note:

1. Do not "cross multiply" or multiply both side with $Q(X)$ because the sign of $Q(X)$ might be positive or negative. Therefore, we do not know whether the symbol of inequality is to be reversed or not.

2. Rewrite the inequalities with 0 on the right and use only addition or subtraction to get an equivalent inequality.

Example 2

Find the solution set of following inequalities:

(a) $\dfrac{x-2}{x+4} \leq 0$

(b) $\dfrac{3x+1}{x+4} \leq 1$

(c) $5x^2 > 3x+2$

Exercises

1. Solve the following linear inequalities :

(a) $6x - 5 > x + 20$ Answer : $x > 5$

(b) $-2 \leq \dfrac{7-3x}{4} < 10$ Answer : (-11,5]

2. Solve $\dfrac{x+3}{x-2} < 5$ Answer : $x < 2 \cup x > \dfrac{13}{4}$

LECTURE 5 OF 6

TOPIC:
2.0 Equation, Inequalities and Absolute Values

SUBTOPIC:
2.3 Absolute Values

LEARNING OUTCOMES:

At the end of the lesson, student should be able to:

(a) Use the properties of absolute values

(b) Solve absolute equations of the forms

(i) $|ax+b|=c$;

(ii) $|ax+b|=cx+d$;

(iii) $|ax+b|=|cx+d|$; and

(iv) $|ax^2+bx+c|=d$.

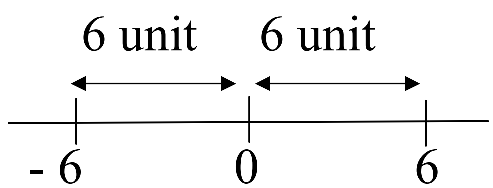
(c) Solve absolute inequalities of the forms $|ax+b|>cx+d$. Also apply to inequality involving notations $<, \leq$ and $\geq$.

CONTENT

Definition

On the real number line, the distance of a number x from 0 (origin) is called the absolute value of x and is denoted by $|x|$.

For example: $|6| = 6$ and $|-6| = 6$ because both 6 and – 6 are 6 units from 0.
(See Fig. below)



Aside from its geometrical interpretation, the absolute value of a can be defined as;

$$|a| = \begin{cases} a, \ if \ \ a \geq 0 \\ -a, \ if \ \ a < 0 \end{cases}$$

We found that if a is positive or 0, then $\sqrt{a^2} = a$
If a is negative, however, we must write $\sqrt{a^2} = -a$, for example $\sqrt{(-2)^2} = -(-2) = 2$

Thus, for a any real number, $\sqrt{a^2} = |a|$

For example

$$|7| = 7$$

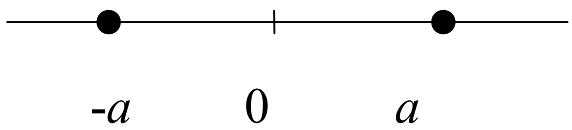
$$|-7| = 7 \text{ because } |-7| = -(-7) = 7$$

Properties of Absolute Values

Properties of absolute values	Examples
1. $\lvert a\rvert \geq 0$	$\lvert 3\rvert = 3 > 0,\ \lvert 0\rvert = 0,\ \lvert -3\rvert = 3 > 0$
2. $\lvert a\rvert = \lvert -a\rvert$	$\lvert 5\rvert = 5$ $\lvert -5\rvert = 5$
3. $\lvert a+b\rvert = \lvert b+a\rvert$	$\lvert -2+6\rvert = \lvert 4\rvert = 4$ $\lvert 6+(-2)\rvert = \lvert 4\rvert = 4$
4. $\lvert a-b\rvert = \lvert b-a\rvert$	$\lvert 3-10\rvert = \lvert -7\rvert = 7$ $\lvert 10-3\rvert = \lvert 7\rvert = 7$
5. $\lvert ab\rvert = \lvert a\rvert\lvert b\rvert$	$\lvert (-3)\times 6\rvert = \lvert -18\rvert = 18$ $\lvert -3\rvert \times \lvert 6\rvert = 3\times 6 = 18$
6. $\left\lvert \frac{a}{b}\right\rvert = \frac{\lvert a\rvert}{\lvert b\rvert}$, $\lvert b\rvert \neq 0$	$\left\lvert \frac{12}{-3}\right\rvert = \lvert -4\rvert = 4$ $\frac{\lvert 12\rvert}{\lvert -3\rvert} = \frac{12}{3} = 4$

Definition $|x| = a, \quad a \geq 0$

$|x| = a$ is equivalent to $x = a$ or $x = -a$



To solve problem such as $|ax + b| = c$, we can use :

(i) Basic Definition, or

(ii) Squaring Both Sides

Example 1

Solve the following equation:

(a) $|x| = 4$

(b) $|x + 8| = 2$

(c) $|5x + 3| = 2x + 9$

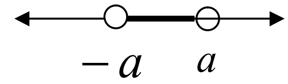
(d) $|2x + 6| = |x - 1|$

(e) $\left|\dfrac{x+1}{x-3}\right| = 2$

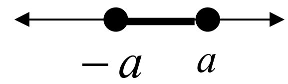
(f) $|x^2 - 6x + 4| = 4$

Absolute value inequalities

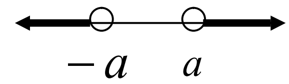
$|x| < a$ is equivalent to $a < x < a$



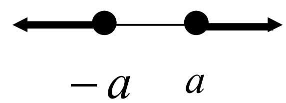
$|x| \le a$ is equivalent to $a \le x \le a$



$|x| > a$ is equivalent to $x > a$ or $x < -a$



$|x| \ge a$ is equivalent to $x \ge a$ or $x \le -a$



There are two methods in solving absolute value inequalities:

1. Basic Definition
2. Squaring Both Sides

Example 2

Solve the following inequalities

(a) $|2x+3| < 5$

(b) $|x-1| \le 4$

LECTURE 6 OF 6

TOPIC:
2.0 Equation, Inequalities and Absolute Values

SUBTOPIC:
2.3 Absolute Values

LEARNING OUTCOMES:

At the end of the lesson, student should be able to solve absolute inequalities of the forms:

(i) $|ax+b| < cx+d$

(ii) $|ax+b| < |cx+d|$

(iii) $\left|\dfrac{ax+b}{cx+d}\right| < e$

(iv) $|ax^2+bx+c| < d$

Also apply to inequality involving notations $<$, $\leq$ and $\geq$.

CONTENT

Example 1

Solve the following inequalities:

(a) $|4x+1| > 2x-5$

(b) $|x^2-5x+3| \leq 3$

Example 2

Solve the inequality $\left|\dfrac{1}{x-2}\right| < 3$

Example 3

Solve $\left|\dfrac{10-2x}{3x+1}\right| > 1$

Example 4

Solve $\left|\dfrac{x+2}{x-3}\right| \geq 3$

Inequality Involving Two Absolute Values

Example 5

Solve $|x-3| \geq |2x+5|$

Example 6

Solve $|x-2| < |2x+3|$

Exercises

1. $|2x-5|>9$ $\quad :(-\infty,-2)\cup(7,\infty)$
2. $|x^2-4x-16|>16$ $\quad :(-\infty,-4)\cup(0,4)\cup(8,\infty)$
3. $|2x+1|<3x+2$ $\quad :\left(-\frac{3}{5},\infty\right)$
4. $|5x+2|>2x-1$ $\quad :(-1,\infty)\cup\left(-\infty,-\frac{1}{7}\right)$
5. $\left|\frac{2x}{x+3}\right|\geq 1$ $\quad :(-\infty,-3)\cup(-3,-1]\cup[3,\infty)$
6. $\left|\frac{3x+4}{x+2}\right|<1$ $\quad :\left(-\frac{3}{2},-1\right)$
7. $|2x+1|>|x-3|$ $\quad :(-\infty,-4)\cup\left(\frac{2}{3},\infty\right)$
8. $|x+2|>3|x-4|$ $\quad :\left(\frac{5}{2},7\right)$