LECTURE 1 OF 5

TOPIC : 1.0 NUMBER SYSTEM

SUBTOPIC : 1.1 Real Numbers

LEARNING OUTCOMES :

At the end of the lesson, students should be able to :

(a) define and understand natural numbers (N), whole numbers (W), integers (Z), prime numbers, rational numbers (Q) and irrational numbers ($\overline{Q}$).

(b) represent rational and irrational numbers in decimal form.

(c) represent the relationship of number sets in a real numbers diagrammatically showing $N \subset W \subset Z \subset Q$ and $Q \cup \overline{Q} = R$.

(d) represent open, closed and half open interval and their representations on the number line.

(e) simplify union, $\cup$, and intersection, $\cap$, of two or more intervals with the aid of number line.

SET INDUCTION:

1.1 Real Numbers

The lecturer will ask the students several questions such as;

i. how many children in the family,

ii. what their heights,

iii. lecturer then give them a situation problem i.e "Assume that a woman wants to buy fishes. The total price of the fishes is RM10, but the woman has only RM5. So she owes RM5 from the seller. (RM5 – RM10 = - RM5).

In conclusion, the lecturer will explain that each answer can be classified into different types of numbers where dates of birth belong to natural numbers, the height is rational numbers and –RM5 is a negative number.

CONTENT

The most fundamental collection or set of numbers is the set of counting numbers or **natural numbers** and denoted by N. Mathematically, this can be written

$$N = \{1, 2, 3 \ldots\}$$

Prime numbers are the natural numbers that are greater than 1 and only can be divided by itself and 1.

$$\text{Prime number} = \{2, 3, 5, 7 \ldots\}$$

The natural numbers, together with the number 0 are called **the whole numbers.** The set of the whole number is written as follow:

$$W = \{0, 1, 2, 3 \ldots\}.$$

The whole numbers together with the negative of counting numbers form the set of **integers** and denoted by Z.

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$$

The set of positive integers is denoted by
$Z^+ = \{1, 2, 3 \ldots\}$ and the set of negative integers is denoted by $Z^- = \{\ldots, -3, -2, -1\}$.
Hence $Z = Z^- \cup \{0\} \cup Z^+$.

Furthermore the elements in Z can be classified as even and odd numbers where

the set of **even numbers** = $\{2k, k \in Z\}$
the set of **odd numbers** = $\{2k + 1, k \in Z\}$

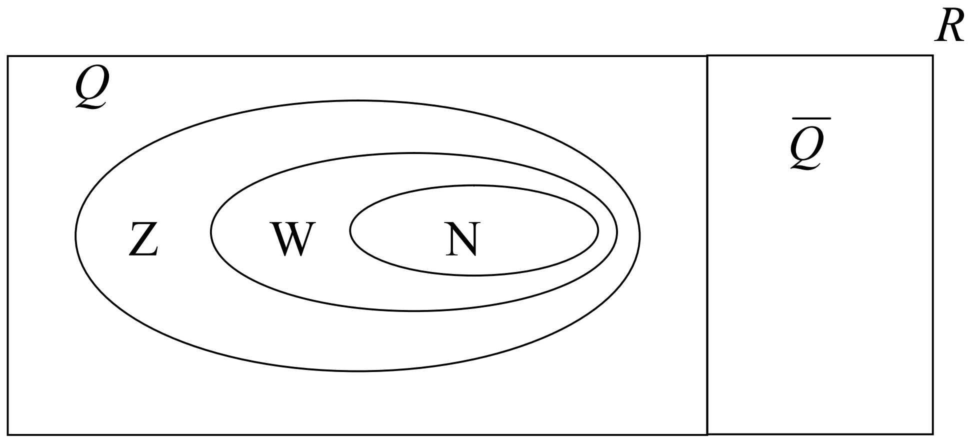
A **rational number** is any number that can be represented as a ratio (quotient) of two integers and can be written as $Q = \left\{ \frac{a}{b} ; a, b \in Z, b \neq 0 \right\}$. Rational numbers can be expressed as terminating or repeating decimals for example 5, 1.5 and –0.333333….
Irrational numbers is the set of numbers whose decimal representations are neither terminating nor repeating. Irrational numbers cannot be expressed as a quotient for example $\sqrt{3}, \sqrt{5}$ and π .

Rational Numbers	**Irrational Numbers**
$\frac{1}{3} = 0.333\ldots$ $\frac{4}{11} = 0.363636\ldots$ $\frac{1}{4} = 0.25$	$\sqrt{2} = 1.41421356\ldots$ $\pi = 3.14159265\ldots$

The real number ($\Re$)

The real number consists of rational numbers and irrational numbers.

Relationship of Number Sets



From the diagram, we can see that :

1. $N \subset W \subset Z \subset Q \subset \Re$

2. $Q \cup \overline{Q} = \Re$

Example 1

For the set of {-5, -3, -1, 0, 3, 8}, identify the set of
(a) natural numbers
(b) whole numbers
(c) prime numbers
(d) even numbers
(e) negative integers
(f) odd numbers

Example 2

Given S = $\{-9,\ \sqrt{7},\ \frac{1}{3},\ \pi^2,\ 0,\ 0.\overline{13},\ 4,\ 5.1212\ldots\}$,

identify the set of

(a) natural numbers

(b) whole numbers

(c) integers

(d) rational numbers

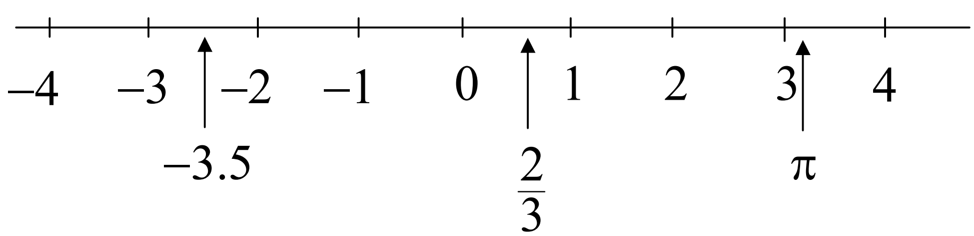
(e) irrational numbers

(f) real numbers

The Number Line

The set of numbers that corresponds to all point on number lines is called the set of **real number.** The real numbers on the number line are ordered in increasing magnitude from the left to the right

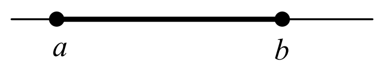
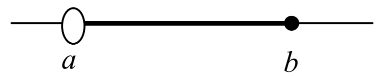
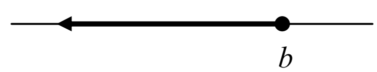
For example for -3.5, $\frac{2}{3}$ and π can be shown on a real number line as



Symbol	Description	Example
$a = b$	*a* equal to *b*	$3 = 3$
$a < b$	*a* less than *b*	$-4 < 4$
$a > b$	*a* greater than *b*	$5 > 0$

Note: The symbols '<' or '>' are called **inequality signs**

All sets of real numbers between a and b, where $a < b$ can be written in the form of intervals as shown in the following table.

Type of Intervals	Notations	Inequalities	Representation on the number line
Closed interval	$[a, b]$	$a \leq x \leq b$	
Open interval	(a, b)	$a < x < b$	
Half-open interval	$(a, b]$	$a < x \leq b$	
Half-open interval	$[a, b)$	$a \leq x < b$	
Open interval	$(-\infty, b)$	$x < b$	
Half-open interval	$(-\infty, b]$	$x \leq b$	
Open interval	(a, ∞)	$x > a$	
Half- open interval	$[a, \infty)$	$x \geq a$	

Note:
The symbol ∞ is not a numerical. When we write $[a, \infty)$, we are simply referring to the interval starting at a and continuing indefinitely to the right.

Example 3

List the number described and graph the numbers on a number line.

(a) The whole number less than 4

(b) The integer between 3 and 9

Example 4

Represent the following intervals on the real number line and state the type of the interval.

(a) $[-1, 4]$

(b) $\{x : 2 < x < 5\}$

(c) $[2, \infty)$

(d) $\{x : x \leq 0, x \in R\}$

Intersection and Union

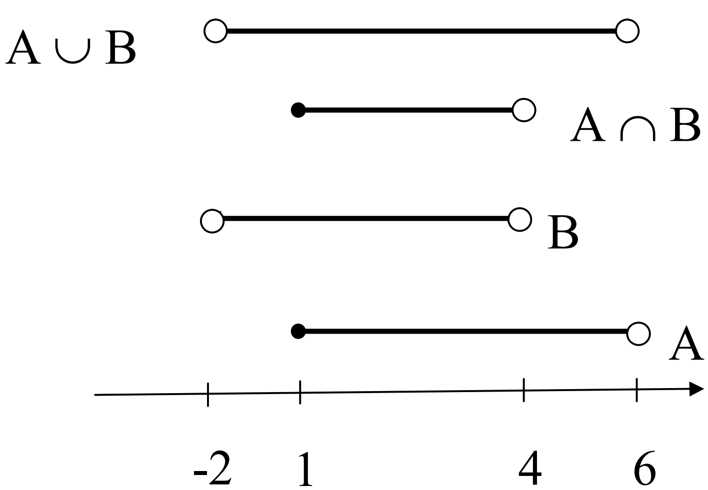
Intersection and union operation can be performed on intervals.

For example,
Let A = [1 , 6) and B = (–2, 4),

Intersection of set A and set B is a half-open interval [1,4).

The union of set A and set B is given by
$A \cup B = (-2, 6)$ is an opened interval.

All these can be shown on a number line given below:



Example 5

Solve the following using the number line:

(a) $[0, 5) \cup (4, 7)$
(b) $(-\infty, 5) \cap [-1, 9\ )$
(c) $(-\infty, 0] \cup [0, \infty)$
(d) $(-4, 2) \cup (0, 4] \cap [-2, 2\ )$

Example 6

Given A = $\{x : -2 < x \leq 5\}$ and B = $\{x : 0 < x \leq 7\}$. Show that $A \cap B = (0, 5]$.

Exercise

1. Given $\{-7, -\sqrt{3}, 0, \frac{1}{5}, \frac{\pi}{2}, \sqrt{4}, 0.16, 0.8181\ldots\}$.
   List the numbers for
   (a) N (b) Z (c) Q (d) R
   (e) $\overline{Q}$

2. Represent these intervals on a number line.
   (a) $[2, 5]$ (b) $(-4, 5]$
   (c) $(-8, 8)$ (d) $[-2, \infty)$

3. Write down the following solution set in an interval notation.
   (a) $\{x : x \geq 6\}$ (b) $\{x : -3 \leq x < 6\}$
   (c) $\{x : -5 < x \leq 5\}$ (d) $\{x : 0 \leq x \leq 4\}$
   (e) $\{x : x \leq 8\} \cup \{x : x \geq 13\}$
   (f) $\{x : x \leq 4\}$

4. Given A = $[2, 5]$, B = $(-3, 5]$, C = $(-7, 7)$ and D = $[-3, \infty)$. Find
   (a) $A \cap B$ (b) $A \cup B$
   (c) $A \cap C \cap D$ (d) $C \cup D \cap A$

Answer

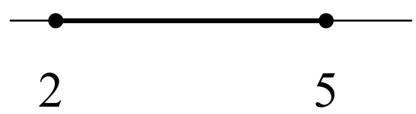
1. (a) none (b) $\{-7, 0, \sqrt{4}\}$

   (c) $\{-7, 0, \frac{1}{5}, \sqrt{4}, 0.16, 0.8181\ldots\}$

   (d) $\{-7, -\sqrt{3}, \sqrt{4}, 0, \frac{1}{5}, \frac{\pi}{2}, 0.16, 0.8181\ldots\}$

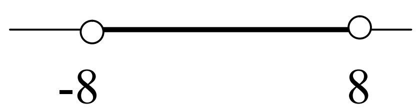
   (e) $\{-\sqrt{3}, \frac{\pi}{2}\}$

2. (a)



   (b)

   (c)



   (d)

4. (a) $[6, \infty)$ (b) $[-3, 6)$

   (c) $(-5, 5]$ (d) $[0, 4]$

   (e) $(-\infty, 8]\cup[12, \infty)$; half- open interval

   (f) $(-\infty, 4]$; half-open interval

5. (a) $[2, 5]$ (b) $(-3, 5]$

   (c) $[2, 5]$ (d) $[2, 5]$

LECTURE 2 OF 5

TOPIC : 1.0 NUMBER SYSTEM

SUBTOPIC : 1.2 Complex Numbers

LEARNING OUTCOMES :

At the end of the lesson, students should be able to :

(a) represent a complex number in Cartesian form.

(b) define the equality of two complex numbers.

(c) determine the conjugate of complex numbers $(\bar{z})$.

(d) perform algebraic operations on complex numbers.

SET INDUCTION:

Look at this equation

$$x^2 + 1 = 0$$

This equation does not have real roots as we cannot find the value for x since $x = \pm\sqrt{-1}$. This problem was encountered by Heron Alexandra. One hundred years later, Mahavira from India stated that a negative value does not have square root because there is no number that can be squared to produce it. In 1637, Descrates of France, introduced ‘real numbers’ and ‘imaginary numbers’. This idea was used by Euler from Switzerland who defined imaginary numbers as real multiples of $\sqrt{-1}$ in 1948. However ‘complex number’ was introduced hundred years later by Gauss from Germany (1832).

CONTENT

We have already seen equation such as $x^2 - 1$ whose roots are complex number ***C***, is a basis of new set of numbers.

The Imaginary Unit *i*

The imaginary unit i is defined as $I = \sqrt{-1}$, where $i^2 = -1$

Using the imaginary unit i, we can express the square root of any negative number as a real multiple of i. For example,

$$\begin{aligned} \sqrt{-16} &= \sqrt{-1 \times 16} \\ &= \sqrt{i^2} \times \sqrt{16} \\ &= i \times 4 \\ &= 4\,i \end{aligned}$$

Power of i can be simplified, therefore

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

Complex numbers (C) consist of two parts, i.e $Re(z)$ which represents the real part and $Im(z)$ represents the imaginary part.

Complex numbers are not ordered and cannot be represented on a real number line. The real number (R) is a subset of complex number (C), i.e $R \subset C$.

Complex numbers can be expressed as a cartesian form $z = x + yi$ where $Re(z) = x$ and $Im(z) = y$.

For example if $z = 2 + 3i$, then $Re(z) = 2$, $Im(z) = 3$

Equality of two complex numbers

Two complex numbers z_1 and z_2 are said to be equal if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Example 1

Find the value of m and n if $z_1 = z_2$ for $z_1 = 3 + 2i$ and $z_2 = m + ni$.

Example 2

Find a and b for the following equations

(a) $a + b + (a - b)i = 6 + 4i$

(b) $a + 2b + (a - b)i = 9$

Algebraic Operations on Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c and d are real numbers, then

i) $z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)\mathrm{i}$

ii) $z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$

iii) $z_1 z_2 = (a + bi)(c + di)$

$$= ac + adi + bc\mathrm{i} + bdi^2$$

$$= (ac - bd) + (ad + bc)i, \quad i^2 = -1$$

Example 3

Given that $z = 2 + 3i$ and $w = 7 - 6i$, find

(a) $z + w$ (b) $w - z$

Example 4

Given that

(a) $z = 4 + 3i$ and $w = 7 + 5i$, find zw.

(b) $u = 4 - 2i$ and $v = -7 + 5i$, find vu.

Conjugates of Complex Numbers

$(a+bi)$ is the conjugate of $(a-bi)$

$(a-bi)$ is the conjugate of $(a+bi)$

For example, $(4-3i)$ is the conjugate of $(4+3i)$.

Conjugate of a complex number, z is denoted by $\bar{z}$ and it can be used to simplify the division of two complex numbers.

Example 5

Simplify the expressions:

(a) $\dfrac{1}{i}$ (b) $\dfrac{3}{1+i}$

(c) $\dfrac{4+7i}{2+5i}$ (d) $\dfrac{2+i}{3-i\sqrt{2}}$

Example 6

If $z = \dfrac{1+i}{2-i}$, find $\bar{z}$ in the Cartesian form $a + bi$.

Example 7

If $z = 1 - 2i$, express $z + \dfrac{1}{z}$ in the form $a + bi$.

Example 8

Solve each of the following equations for the complex number z.

(a) $4+5i=z-(1-i)$ (b) $(1+2i)z=2+5i$

Example 9

Solve $(x+yi)(3-i)=1+2i$ where x and y are real.

Exercise:

1. Write the following complex numbers in the form $x + yi$.

   (a) $(3 + 2i) + (2 + 4i)$

   (b) $(4 + 3i) - (2 + 5i)$

   (c) $(3 + 2i)(4 - 3i)$

Ans: (a) $5 + 6i$

(b) $2 - 2i$

(c) $18 - i$

2. Solve for z when

   (a) $z(2 + i) = 3 - 2i$

   (b) $(z + i)(1 - i) = 2 + 3i$

   (c) $\frac{1}{z} + \frac{1}{2-i} = \frac{3}{1+i}$

Ans : (a) $\frac{4}{5} - \frac{7}{5}i$ (b) $-\frac{1}{2} + \frac{3}{2}i$

(c) $\frac{11}{14} + \frac{17}{14}i$

3. Find the values of the real numbers x and y in each of the following:

(a) $\dfrac{x}{1+i}+\dfrac{y}{1-2i}=1$

(b) $\dfrac{x}{2-i}+\dfrac{yi}{i+3}=\dfrac{2}{1+i}$

Ans : (a) $x=\dfrac{4}{3}, y=\dfrac{5}{3}$

(b) $x=4, y=-6$

LECTURE 3 OF 5

TOPIC : 1.0 NUMBER SYSTEM

SUBTOPIC : 1.2 Complex Numbers

LEARNING OUTCOMES :

At the end of the lesson, students should be able to :

(e) represent a complex number in polar form

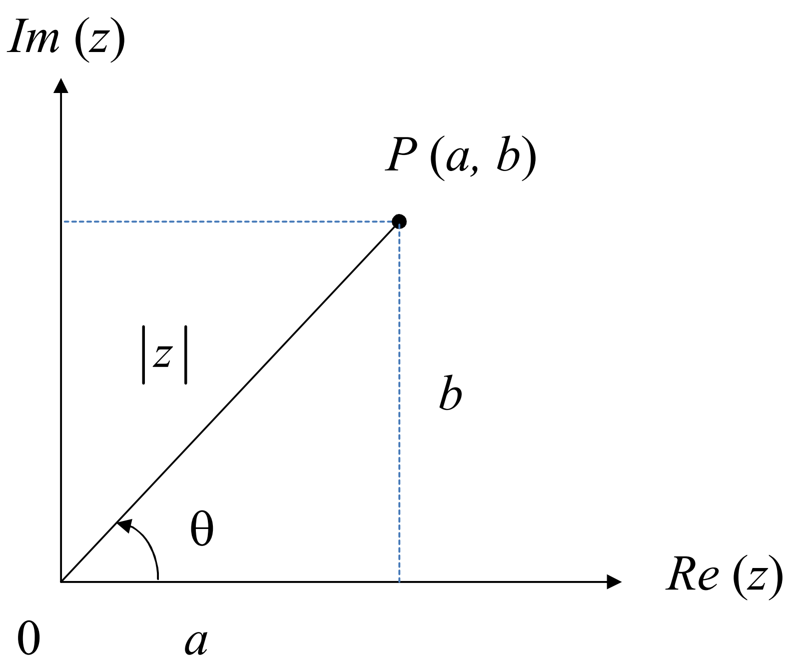
$z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta < \pi$.

Content

Argand Diagram and Modulus

Complex number $z = a + bi$ can be represented in a plane as a point $P(a, b)$. the x-coordinate represents the real part of the complex number while the y-coordinate represents the imaginary part of a complex number.

The plane on which a complex number is represented is called the complex number plane and the figure represented by the complex numbers as points in a plane is known as an Argand Diagram. The diagram below shows the complex number $z = a + bi$ represented as a point $P(a, b)$ in the Argand diagram.



The modulus of a complex number $z = a + bi$ is defined as the distance from the origin to the point $P(a, b)$.

Modulus z = length OP = r = $|z| = \sqrt{a^2+b^2}$

Argument of z = arg (z) = θ = $\tan^{-1}\left(\dfrac{b}{a}\right),\ -\pi<\theta\leq\pi$,

which is called the principal argument.
Argument of z is the angle measured from the positive x-axis, either in an anti clockwise or clockwise manner.

Example 1

Represent the following complex numbers on argand diagram and find its modulus and argument.

(a) $z=2+4i$
(b) $z=2-4i$

Polar Form of Complex Number

Let $z = a + bi$, where $a, b \in \Re$, be represented by the point $P(a, b)$. from diagram we know that

$$a = r\cos\theta \quad \text{---(1)}$$
$$b = r\sin\theta \quad \text{---(2)}$$

Then, $z = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta)$

Polar form : $z = r(\cos\theta + i\sin\theta)$ is called the polar form of a complex number z.

Example 2

Write the following complex numbers in polar forms.

(a) $-2 + 2i$

(b) $\sqrt{3} + i$

(c) $1 - \sqrt{3}i$

LECTURE 4 OF 5

TOPIC **:** 1.0 NUMBER SYSTEM

SUBTOPIC **:** 1.3 Indices, Surd and Logarithms

LEARNING OUTCOMES :
At the end of the lesson, student should be able to:

(a) state the rules of indices

(b) explain the meaning of a surd and its conjugate and to carry out algebraic operations on surds

SET INDUCTION:

The words 'square' and 'cube' come from geometry. If a square has side x units, the square of the number x gives the area of the square. If a cube has side x units, the cube of the number x gives the volume of the cube. These are written as x^2 and x^3 .This notation can be extended to higher powers such as x^4 and x^5, although there is no obvious physical interpretation of these expressions. The index notation is used to save us having to write several multiplications.

5^6 means $5\times5\times5\times5\times5\times5$

As defined, the notation is restricted to positive whole numbers, but it can be further extended to negative and fractional powers, so that we can speak of 2^{-3}, or of $3^{\frac{1}{2}}$.

CONTENT:

We can write $2\times2\times2\times2\times2$ as 2^5. This is read as "two to the power of five" where 2 is the base and 5 is the power. In general $\underbrace{a\times a\times a\times ....\times a}_{n\ times} = a^n$ where a is called the base and n is the index or power. Sometimes this is read as "a is raised to the power of n".

Rules of indices

1. $a^m \times a^n = a^{m+n}$
2. $\left(a^m\right)^n = a^{mn}$
3. $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$
4. $a^0 = 1$ provided $a \neq 0$
5. $a^{-m} = \dfrac{1}{a^m}$
6. $a^{\frac{1}{m}} = \sqrt[m]{a}$, m^{th} roots of a
7. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(a^{\frac{1}{n}}\right)^m$
8. $\left(ab\right)^n = a^n b^n$
9. $\left(\dfrac{a}{b}\right)^n = \left(\dfrac{a^n}{b^n}\right)$

There are some special cases to consider.

(i) Division when powers are equals (Zero index)

$$6^3 \div 6^3 = 6^{3-3}$$
$$1 = 6^0$$

This result leads to another rule.

$$\boldsymbol{a^0 = 1} \text{ provided } a \neq \mathbf{0}$$

(ii) Division when the power of the denominator is greater than that of the numerator (Negative index)

$$7^3 \div 7^5 = 7^{3-5}$$
$$\frac{7x7x7}{7x7x7x7x7} = 7^{-2}$$
$$\frac{1}{7^2} = 7^{-2}$$

$$a^{-m} = \frac{1}{a^m}$$

A negative power indicates a reciprocal.

(iii) Rational index

Rule 1 can be used to introduce rational power.

Suppose that $p = q = \frac{1}{2}$ in rule 1

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}}$$

$$\left(a^{\frac{1}{2}}\right)^2 = a^1 = a$$

$$a^{\frac{1}{2}} = \sqrt{a}$$

The meaning of power $\frac{1}{3}$ can be established in a similar way.

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$

$$\left(a^{\frac{1}{3}}\right)^3 = a^1 = a$$

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

This means 'power $\frac{1}{3}$' implies a cube root.

Investigate powers $\frac{1}{4}, \frac{1}{5}$, and so on in the same way. Notice how the fraction is related to the root.

$$\boxed{a^{\frac{1}{m}} = \sqrt[m]{a}}$$

Example 1

Simplify

(a) $\dfrac{3^5 \times 3^6}{3^4}$

(b) $\dfrac{18x^2y^5}{3x^4y}$

Example 2:

Without using calculator, evaluate:

(a) $9^{-\frac{3}{2}}$

(b) $\left(1\dfrac{11}{25}\right)^{-\frac{1}{2}}$

Example 3

Simplify the following expression

(a) $\left(\dfrac{a^2b^{-3}}{x^{-1}y^2}\right)^3\left(\dfrac{x^{-2}b^{-1}}{a^{\frac{3}{2}}y^{\frac{1}{3}}}\right)$

(b) $\dfrac{2^{2n+4}-24.2^{2(n-1)}}{10\left(2^n\right)^2}$

SURDS

Rules of Surds

1. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $a, b \geq 0$

2. $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$, $a \geq 0$, $b > 0$

3. $a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$

4. $a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b}$

5. $\sqrt{a} \times \sqrt{a} = a$

6. $\sqrt{a} + \sqrt{a} = 2\sqrt{a}$

7. $\sqrt{a} \div \sqrt{b} = \sqrt{\dfrac{a}{b}}$

8. $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$

9. $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$

Remark:

$$\sqrt{a+b} \neq \left(\sqrt{a} + \sqrt{b}\right)$$

Example 4

Simplify:

(a) $\sqrt{45}$

(b) $\sqrt{24}$

(c) $6\sqrt{7}+2\sqrt{7}$

(d) $5\sqrt{3}-\sqrt{27}$

Multiplying Radicals

There are 2 types.

1. $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$
2. $a\sqrt{b}\times c\sqrt{d} = ac\sqrt{bd}$

Example 5

Multiply

(a) $3\sqrt{6} \times 5\sqrt{7}$

(b) $8\sqrt{2}\left(5\sqrt{6} + \sqrt{2}\right)$

(c) $\left(2\sqrt{3} + 4\sqrt{2}\right)\left(6\sqrt{3} + 2\sqrt{2}\right)$

Rationalising the denominator

Rationalising the denominator is an algebraic operation to change the denominator to a rational number. To do so, we multiply both the numerator and the denominator of the fraction by the conjugate of the denominator of the fraction.

If Denominator **Contains the Factor**	**Multiply by**	*To Obtain Denominator* **Free from surds**
$\sqrt{3}$	$\sqrt{3}$	$\left(\sqrt{3}\right)^2 = 3$
$\sqrt{3}+1$	$\sqrt{3}-1$	$\left(\sqrt{3}\right)^2 - 1^2 = 3-1=2$
$\sqrt{2}-3$	$\sqrt{2}+3$	$\left(\sqrt{2}\right)^2 - 3^2 = 2-9=-7$
$\sqrt{5}-\sqrt{3}$	$\sqrt{5}+\sqrt{3}$	$\left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2 = 5-3=2$

Example 6

Rationalise the denominator of the following fractions:

(a) $\dfrac{5}{\sqrt{3}}$

(b) $\dfrac{1}{7-\sqrt{2}}$

(c) $\dfrac{2\sqrt{3}}{5-\sqrt{3}}$

Example 7

Rationalise the denominator and simplify.

(a) $\dfrac{\sqrt{17}-\sqrt{5}}{\sqrt{17}+\sqrt{5}}$

(b) $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}+\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$

(c) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\sqrt{2}$

Exercise:

1. Find the value of each of the following

   a) $64^{\frac{2}{3}}$ b) $\left(\frac{27}{8}\right)^{\frac{-1}{3}}$

   c) $(0.04)^{\frac{-3}{2}}$ d) $2^n \times 8^{3n} \div 4^{5n}$

2. Simplify

   a) $\frac{\sqrt{63}}{3}$

   b) $\sqrt{75} + 2\sqrt{48} - 5\sqrt{12}$

3. Expand and simplify:

   a) $(2 - 3\sqrt{3})(3 + 2\sqrt{3})$

   b) $(5 - 2\sqrt{7})(5 + 2\sqrt{7})$

Answer :

1. a) 16   b) $\frac{2}{3}$

   c) 125   d) 1

2. a) $\sqrt{7}$ b) $3\sqrt{3}$

3. a) $-12 - 5\sqrt{3}$ b) -3

LECTURE 5 OF 5

TOPIC : 1.0 NUMBER SYSTEM

SUBTOPIC : 1.3 Indices, Surds and Logarithms

LEARNING OUTCOMES :

At the end of the lesson, students should be able to :

(c) state the laws of logarithm such as

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(iii) $\log_a M^N = N \log_a M$

(d) change the base of logarithm using $\log_a M = \dfrac{\log_b M}{\log_b a}$

Set Induction

John Napier(1550-1617), a Scottish mathematician, invented the logarithms in 1614 with a specific purpose of reducing the amount of work involved in multiplying and dividing large numbers. Even with the advent of computers and calculators, logarithms have not lost their significance, it is still an important tool in mathematics. In this chapter we discuss what logarithms are and how they can be used.

Logarithms

If a, x and n are related in such a way that $a^x = n$ then x is said to be the logarithm of the number n which respect the base a, i.e $\log_a n = x$

$$\boxed{\log_a n = x \iff a^x = n}$$

Example 1

a) $\log_2 8 = 3 \iff 2^3 = 8$

b) $10^2 = 100 \iff \log_{10} 100 = 2$

Example 2

For each of the following, write down an expression for a logarithm in a suitable base:

(a) $1000 = 10^3$

(b) $\dfrac{1}{32} = \dfrac{1}{2^5}$

Example 3

Without using calculator, find the value of:

(a) $\log_2\left(\dfrac{1}{8}\right)$ (b) $\log_{\frac{1}{2}}(4)$

Natural Logarithms

The natural logarithmic function and the natural exponential function are inverse functions of each other.

The symbol $\ln x$ is an abbreviation for $\log_e x$, and we refer to it as the **natural logarithms of *x***.

Definition of Common Logarithms	**Definition of Natural Logarithms**
$\log x = \log_{10} x$, for every $x > 0$	$\ln x = \log_e x$, for every $x > 0$

The Laws of Logarithms

Let $\log_a M = x$ and $\log_a N = y$

Therefore $a^x = M$ and $a^y = N$

$$a^x . a^y = MN$$

or $a^{x+y} = MN$

therefore $x + y = \log_a MN$

i.e $\boxed{\log_a M + \log_a N = \log_a MN}$ **RULE 1**

Similarly $a^x \div a^y = \dfrac{M}{N}$

or $a^{x-y} = \dfrac{M}{N}$

therefore $x - y = \log_a \left(\dfrac{M}{N}\right)$

i.e $\boxed{\log_a M - \log_a N = \log_a \left(\dfrac{M}{N}\right)}$ **RULE 2**

Let $\log_a M^N = z$

Therefore $a^z = M^N$

So $a^{\frac{z}{N}} = M$

Therefore $\log_a M = \dfrac{z}{N}$

Or $N \log_a M = z$

i.e $\boxed{\boldsymbol{N \log_a M = \log_a M^N}}$ **RULE 3**

The rule 1,2 and 3 are known as the three basic laws of logarithms, i.e.

1. product Rule : $\boldsymbol{\log_a M + \log_a N = \log_a MN}$

   e.g. $\ln 4x = \ln 4 + \ln x$

2. Quotient Rule : $\boldsymbol{\log_a M - \log_a N = \log_a \left(\frac{M}{N}\right)}$

   e.g. $\ln \dfrac{3}{7} = \ln 3 - \ln 7$

3. Power Rule : $\boldsymbol{N \log_a M = \log_a M^N}$

   e.g. $\ln \sqrt{x} = \ln x^{\frac{1}{2}} = \dfrac{1}{2} \ln x$

The following table lists the general properties for natural logarithmic form.

Logarithms with base *a*	**Common logarithms**	**Natural logarithms**
1) $\log_a(1) = 0$	$\log 1 = 0$	$\ln 1 = 0$
2) $\log_a a = 1$	$\log 10 = 1$	$\ln e = 1$
3) $\log_a a^x = x$	$\log 10^x = x$	$\ln e^x = x$
4) $a^{\log_a(x)} = x$	$10^{\log(x)} = x$	$e^{\ln(x)} = x$

Example 4

Given $\log 2 = 0.301$ and $\log 6 = 0.778$.

Hence, find $\log 12$.

Example 5

Given $\log 6 = 0.778$, find $\log 36$.

Example 6

Write the following as single logarithms:

(a) $\log 8 - \log 6 + \log 9$

(b) $2 \ln (x + 7) - \ln x$

Properties of Logarithms

The properties of logarithms are very similar to the properties of exponents because logarithms are exponents. The properties will be used in solving logarithmic equations.

Properties of logarithms	Example
1) $\log_a a = 1$	$\log_5(5) = 1$
2) $\log_a(1) = 0$	$\log_2(1) = 0$
3) $\log_a(a^m) = m$	$\log(10^7) = 7$
4) $a^{\log_a(m)} = m$	$5^{\log_5(3)} = 3$
5) $\log_a\left(\frac{1}{N}\right) = -\log_a(N)$	$\log\left(\frac{1}{3}\right) = \log 1 - \log 3$ $= -\log 3$
6) $\log_a(m) = \log_a(n)$ $\Rightarrow m = n$	$\log x = \log 4 \Rightarrow x = 4$

Change of Base

Tables are not readily available which list the values of expressions such as $\log_7 2$.

However if $\log_7 2 = x$

Then $7^x = 2$

So $x \log 7 = \log 2$

Or $x = \dfrac{\log 2}{\log 7}$

Therefore we have changed the base from 7 to 10 and can now use our ordinary tables or calculator to evaluate $\log_7 2$.

Therefore $\log_7 2 = \dfrac{\log 2}{\log 7} = \dfrac{0.3010}{0.8451} = 0.3562$

A general formula for changing from base a to base b can be derived in the same way.

If $\log_a M = x$

Then $M = a^x$

So $\log_b M = x \log_b a$

$$x = \frac{\log_b M}{\log_b a}$$

or $$\boxed{\log_a M = \frac{\log_b M}{\log_b a}}$$

in the special case when $M = b$, this identity becomes

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$$

Example 7

Evaluate correct to four decimal places.

(a) $\log_3 5$

(b) $\log_5 10$

Example 8

Find x if $\ln x = 4.7$

Exercise

1. Without using calculator, find the values of:

   (a) $\log 1000$ (b) $\log 0.01$

2. Find the values of the following, to three significant figures:

   (a) $\log 50$ (b) $\log(\frac{1}{3})$

3. Without using calculator, find the exact values of x in:

   (a) $x = \log_2 64$ (b) $x = \log_3 (\frac{1}{81})$

4. Find the values of the a

   (a) $3 = \log_a 125$

   (b) $-2 = \log_a (\frac{1}{25})$

   (c) $\frac{1}{2} = \log_a 25$

5. Express y in terms of x for each of the following equations

(a) $\log y = 2 \log x$ ($y = x^2$)

(b) $1 + \log y = 3 \log x$ ($y = \frac{x^3}{10}$)

6. Write the following as single logarithms:

(a) $\log_a x^2 + 3\log_a x - 2\log_a 4x$ ($\log_a \left(\frac{x^2}{16}\right)$)

7. Find the value of x if

(a) $7^x = 8$ ($x = 1.0686$)

(b) $5^{2x} = 8$ ($x = 0.6460$)

(c) $3^{x+1} = 4^{x-1}$ ($x = 8.6336$)